An Automata Theoretic Approach to Branching Time Model Checking

Acknowledgements: after Arie Gurfinkel's notes

Automata and Logic

- There is an intimate connection between automata and logic
- Logic
  - a temporal logic formula φ is identified with all models that satisfy it
- Automata
  - a language of an automaton is the set of all words accepted by it
- The language of an automaton for a logical formula φ is the set of all models that satisfy φ
  - strings for linear logic
  - trees for branching logic

Automata-Theoretic Approach

- Automata-theoretic approach gives a uniform solution to both satisfiability and model-checking
- For a given logical formula φ and a model K
  - φ is satisfiable iff there exists a model that satisfies φ
    - $\Box p$ is satisfiable
    - $\Box(p \wedge \neg p)$ is not
  - model-checking is deciding if φ is satisfied by a given model
- Automata-theoretic solution
  - build an automaton A_φ for the formula φ
    - φ is satisfiable iff A_φ is non-empty
  - combine $A_{\neg\varphi}$ and K into an automaton $A_{\neg\varphi,K}$
    - $K \models \varphi$ iff $A_{\neg\varphi,K}$ is empty

Automata-Theoretic Approach

- Automata provide a clean separation between logic and algorithms
- Constructing an automaton for a formula
  - what does that mean for a model to satisfy the formula
- Solving non-emptiness problem for an automaton
  - how to decide if a given model satisfies the formula

Outline

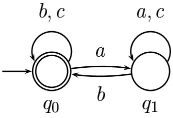
- Automata on infinite words
  - refresher
  - acceptance conditions
  - computational tree of an automaton
  - alternation – a powerful extension of nondeterminism
- Constructing an Alternating Word Automaton for LTL
- Automata on infinite trees
  - deterministic automata
  - nondeterministic automata
  - alternating automata
- Constructing an Alternating Tree Automaton for CTL

Finite-state Automata

- A finite state automaton A is a tuple $(\Sigma, Q, \delta, q_0, \mathcal{F})$, where
  - Σ is a *finite* alphabet
  - Q is a *finite* set of states
  - $\delta \subseteq Q \times \Sigma \times Q$ is a transition relation
  - $q_0 \in Q$ is a designated initial state
  - $\mathcal{F} \subseteq Q^\omega$ is an acceptance condition
- A D-labeled infinite string s is a function $\mathbb{N} \to D$
- A Σ-word w is a Σ-labeled infinite string
  - $w = ababaaa^\omega$
  - $w(0) = a$, $w(1) = b$, $w(3) = a$, etc.

Finite-state Automata

- A run r of an automaton over a word w is a $N \times Q$ labeled string, where
  - a node of r labeled with (n, q) indicates that the automaton reads letter n of w while at state q



state	$\delta(q, a)$	$\delta(q, b)$	$\delta(q, c)$
q_0	q_1	q_0	q_0
q_1	q_1	q_0	q_1

  - a run on $w = aba^\omega$ is

$$(0, q_0), (1, q_1), (2, q_0), (3, q_1), (4, q_1), (5, q_1), \ldots$$

Infinite Occurrences

- $\exists^\omega i \cdot Y(i)$ – there exists infinitely many ith such that $Y(i)$
- For $\rho \in Q^\omega$
  - $In(\rho)$ is the set of states that occur infinitely often
    - $In(\rho) = \{q \in Q \mid \exists^\omega i \cdot \rho(i) = q\}$
- Büchi condition
  - $\mathcal{F}$ is $F \subseteq Q$
  - $In(w) \cap F \neq \emptyset$
  - weak fairness – something occurs infinitely often
- Muller condition
  - $\mathcal{F}$ is $\{F_1, \ldots, F_n\} \subseteq 2^Q$
  - $\exists i \cdot In(w) = F_i$

Acceptance Conditions

- Rabin condition ("pairs")
  - $\mathcal{F}$ is $\{(R_1, G_1), \ldots (R_n, G_n)\}$ with $R_i, G_i \subseteq Q$
  - $\exists i \cdot In(w) \cap R_i = \emptyset \wedge In(w) \cap G_i \neq \emptyset$
  - Rabin $(\emptyset, F)$ is equivalent to Büchi F
- Street condition ("complemented pairs")
  - $\mathcal{F}$ is $\{(F_1, E_1), \ldots, (F_n, E_n)\}$ with $E_i, F_i \subseteq Q$
  - $\forall i \cdot In(w) \cap F_i \neq \emptyset \Rightarrow In(w) \cap E_i \neq \emptyset$
  - strong fairness
    - if infinitely often enabled, then infinitely often executed
    - Street (Q, F) is equivalent to Büchi F

Acceptance Conditions

- Parity condition
  - $\mathcal{F}$ is $F_1 \subseteq \cdots \subseteq F_n$ with $F_i \subseteq Q$
  - smallest i for which $In(w) \cap F_i \neq \emptyset$ is even
- co-Büchi condition
  - $\mathcal{F}$ is $F \subseteq Q$
  - accepts w if $In(w) \cap F = \emptyset$
- Nondeterministic Büchi-, Muller-, Rabin-, and Street-automata all recognize the same ω-languages

Example: Acceptance

- Language over $\{a, b, c\}^\omega$
  - if a occurs infinitely often, then so does b
- Automaton with states q_a, q_b, and q_c, and δ

state	$\delta(q, a)$	$\delta(q, b)$	$\delta(q, c)$
q_a	q_a	q_b	q_c
q_b	q_a	q_b	q_c
q_c	q_a	q_b	q_c

- Acceptance conditions
  - Street – single pair $(\{q_a\}, \{q_b\})$
  - Muller – all states F where $q_a \in F \Rightarrow q_b \in F$
    $\{q_b\}, \{q_c\}, \{q_b, q_c\}, \{q_a, q_b\}, \{q_a, q_b, q_c\}$

Example: Acceptance

- Automaton with states q_a, q_b, and q_c, and δ

state	$\delta(q, a)$	$\delta(q, b)$	$\delta(q, c)$
q_a	q_a	q_b	q_c
q_b	q_a	q_b	q_c
q_c	q_a	q_b	q_c

- Acceptance conditions
  - Rabin
    - either b occurs infinitely often, or both a and b have finite occurrences
    - two pairs $(\emptyset, \{q_b\}), (\{q_a, q_b\}, \{q_c\})$
  - Parity
    - $\emptyset, \{q_b\}, \{q_a, q_b\}, \{q_a, q_b, q_c\}$

Example: Acceptance

- For Büchi acceptance condition simulate Rabin pairs by nondeterminism

state	$\delta(q,a)$	$\delta(q,b)$	$\delta(q,c)$
q_a	q_a	q_b	$\{q_c, q'\}$
q_b	q_a	q_b	$\{q_c, q'\}$
q_c	q_a	q_b	$\{q_c, q'\}$
q'			q'

- Every time c occurs, guess that a suffix containing only c is reached
- Büchi acceptance condition
  - $F = \{q_b, q'\}$

Computational Tree of an Automaton

- A set of all runs of an automaton A over a fixed word w is called a computational tree
- Each node in the computational tree is labeled by a history $h \in Q^*$
- A history is a list of all states visited by the automaton so far
- For a deterministic automaton, the computational tree is linear
  - there is only one possible run!

Computational Tree: Example

- A deterministic automaton over $\{a, b\}$

state	$\delta(q, a)$	$\delta(q, b)$
q_0	q_0	q_1
q_1	q_0	q_1

- Computational tree over $(aab)^\omega$

$$(q_0), (q_0, q_0), (q_0, q_0, q_0), (q_0, q_0, q_0, q_1), \ldots$$

Comp. Tree: Nondeterministic Case

- For a nondeterministic automaton, the computational tree contains all possible choices
- Formally, the computational tree T of A over w is recursively defined as
  - the root is labeled by q_0,
  - for a node $k \in T$ labeled with the history $x \cdot y$, where $x \in Q^*$, and $y \in Q$
    - if $\delta(y, w(|x \cdot y|)) = \{t_1, \dots t_n\}$, then
    - k has n successors, and
    - ith successor is labeled with $x \cdot y \cdot t_i$

Example: nondeterministic automaton

- Nondeterministic automaton over $\{a, b\}$

state	$\delta(q,a)$	$\delta(q,b)$
q_0	$\{q_0, q_1\}$	q_0
q_1	q_1	q_2
q_2	q_2	q_2

  - acceptance condition is Büchi $F = \{q_1\}$
  - corresponds to $\diamond\Box a$
- Computational tree over aba^ω

Computational Tree: Acceptance

- An infinite history $h \in Q^\omega$ corresponds to an infinite branch β of the computational tree iff for any prefix of h there exists a node in β labeled with it
- An automaton A accepts a word w iff
  - there exists an infinite branch β in the computational tree of A over w, such that
    - an infinite history corresponding to β is an accepting run

Alternation

- For a non-deterministic automaton A, a transition $\delta(q, a) = \{t_1, \ldots, t_n\}$ can be interpreted as
  - when A is in state q and has read letter a
    - create n copies of A
    - switch ith copy to state t_i
    - run each copy on the rest of the word
  - a word is accepted iff it is accepted by at least one copy
- We can dualize the acceptance condition to be
  - a word is accepted iff it is accepted by *all* copies
- In this case, the computational tree is linear
  - but, each node is labeled with multiple histories

Example of a Dual Automaton

- Automaton over $\{a, b\}$

state	$\delta(q, a)$	$\delta(q, b)$
q_0	$\{q_0, q_1\}$	q_0
q_1	q_1	q_2
q_2	q_2	q_2

  - just as before but $\{q_0, q_1\}$ means pick both, not pick one!
  - acceptance condition is Büchi $F = \{q_1\}$
- Computational tree over aba^ω is linear

Another Example of a Dual Automaton

- Automaton over $\{a, b, c\}$

state	$\delta(q, a)$	$\delta(q, b)$	$\delta(q, c)$
q_0	$\{q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_1, q_2\}$
q_1	q_3	q_1	q_1
q_2	q_2	q_3	q_2
q_3	q_3	q_3	q_3

  - acceptance condition is Büchi $F = \{q_3\}$
  - accepts $\circ((\diamond a) \wedge (\diamond b))$
- Computational tree over $ccabc^\omega$ is linear

Alternating Automata

- Alternating automata combine the two interpretations
  - the transition relation becomes $Q \times \Sigma \to 2^{2^Q}$
  - $\delta(q, a) = \{T_1, \ldots, T_n\}$ is interpreted as
    - when a is read at state q, pick one of $T_i \subseteq Q$
    - create as many copies of A as $|T_i|$, and send them along the word
    - a word is accepted iff it is accepted by all the copies
- A computational tree of an alternating automaton is
  - branching
  - each node can be labeled with multiple histories

Alternating Automata: Example

- Example alternating automaton over $\{a, b, c\}$

state	$\delta(q, a)$	$\delta(q, b)$	$\delta(q, c)$
q_0	$\{\{q_0\}, \{q_1\}\}$	q_2	$\{q_1, q_2\}$
q_1	q_1	q_3	q_1
q_2	q_3	q_2	q_2
q_3	q_3	q_3	q_3

- Büchi acceptance $\{q_3\}$
- computational tree over $acbaa^\omega$
- a run over $acbaa^\omega$
- corresponds to $a\,\mathcal{U}\,(\diamond a \wedge \diamond b)$

Alternating Automata: Acceptance

- A word is accepted iff there exists an infinite branch such that all of its infinite histories satisfy the acceptance condition
- Alternatively,
  - a run of an alternating word automaton is a tree
    - each branch in the computational tree is a run
    - the set of infinite histories associated with a branch forms a tree
  - a run is accepting iff all of its branches are accepting
  - a word is accepted iff there exists an accepting run

Symbolic Representation

- A transition relation $Q \times A \to 2^{2^Q}$ can be represented symbolically as a boolean formula over Q
  - $q_1 \vee q_2$ is equivalent to $\{\{q_1\}, \{q_2\}\}$
  - $q_1 \wedge q_2 \vee q_3$ is equivalent to $\{\{q_1, q_2\}, \{q_3\}\}$
- Intuition
  - $q_1 \vee q_2$ means
    - split into two copies
    - one switches to q_1, the other to q_2
    - accept iff at least one copy accepts
  - $(q_1 \vee q_2) \wedge q_3$
    - split into 3 copies
    - 1st switches to q_1, 2nd to q_2, and 3rd to q_3
    - accept if both the 3rd copy *and* either one of 1st or 2nd accept

Why Do We Need This?

- Complementation is easy
  - let φ be a boolean formula over X
  - a dual φ_c of φ is obtained by switching $\wedge$ with $\vee$
    - a dual of $(a \wedge b) \vee c$ is $(a \vee b) \wedge c$
  - a complement of $A = (\Sigma, Q, \delta, q_0, \mathcal{F})$ is $A_c = (\Sigma, Q, \delta_c, q_0, \mathcal{F}_c)$, where
    - δ_c is the dual of δ, $\mathcal{F}_c = Q^\omega \setminus \mathcal{F}$
- There is an easy translation from temporal logic (LTL) to alternating Büchi word automaton

From LTL to Automata

- For an LTL formula φ
  - closure of φ, $cl(\varphi)$, is the set of all subformulas of φ
- An alternating automaton A_φ that accepts all 2^{AP} labeled words that satisfy φ is built as follows
  - $A_\varphi = (2^{AP}, cl(\varphi), \delta, \varphi, F)$
  - $\delta(q, \sigma)$ is defined as follows

$$
\begin{array}{lcllcl}
\delta(a,\sigma) & = & a \in \sigma & \delta(\neg a,\sigma) & = & a \notin \sigma \\
\delta(\circ\varphi,\sigma) & = & \varphi & \delta(\Box\varphi,\sigma) & = & \delta(\varphi,\sigma) \wedge \Box\varphi \\
\delta(\diamond\varphi,\sigma) & = & \delta(\varphi,\sigma) \vee \diamond\varphi & \delta(\varphi\,\mathcal{U}\,\psi,\sigma) & = & \delta(\psi,\sigma)\vee \\
 & & & & & \delta(\varphi,\sigma) \wedge \varphi\,\mathcal{U}
\end{array}
$$

  - $F = \{\Box\psi \mid \Box\psi \in cl(\varphi)\}$ is a Büchi acceptance condition

F := F \union { true }

Examples

- $a \,\mathcal{U}\, b$

state	$\delta(q, \{a, b\})$	$\delta(q, \{a\})$	$\delta(q, \{b\})$	$\delta(q, \emptyset)$
a	**true**	**true**	**false**	**false**
b	**true**	**false**	**true**	**false**
$a \,\mathcal{U}\, b$	**true**	$a \,\mathcal{U}\, b$	**true**	**false**

  - no accepting states

Examples

- $a\,\mathcal{U}\,\Diamond b$

state	$\delta(q,\{a,b\})$	$\delta(q,\{a\})$	$\delta(q,\{b\})$	$\delta(q,\emptyset)$
a	**true**	**true**	**false**	**false**
b	**true**	**false**	**true**	**false**
$\Diamond b$	**true**	$\Diamond b$	**true**	$\Diamond b$
$a\,\mathcal{U}\,\Diamond b$	**true**	$(\Diamond b) \vee (a\,\mathcal{U}\,\Diamond b)$	**true**	$\Diamond b$

- no accepting states

Examples

- $\Box a$

state	$\{a\}$	$\emptyset$
a	**true**	**false**
$\Box a$	$\Box a$	**false**

- acceptance condition $\{\Box a\}$
- $(\Box a) \wedge (\Box b)$

state	$\delta(q, \{a, b\})$	$\delta(q, \{a\})$	$\delta(q, \{b\})$	$\delta(q, \emptyset)$
a	**true**	**true**	**false**	**false**
b	**true**	**false**	**true**	**false**
$\Box a$	$\Box a$	$\Box a$	**false**	**false**
$\Box b$	$\Box b$	**false**	$\Box b$	**false**
$(\Box a) \wedge (\Box b)$	$(\Box a) \wedge (\Box b)$	**false**	**false**	**false**

Automata over Infinite Trees

Outline

- Automata on infinite trees
  - deterministic automata
  - nondeterministic automata
  - alternating automata
- Constructing an Alternating Tree Automaton for CTL

Trees

- A tree is a tuple (V_t, V_l, E, r), where
  - V_t and V_l is the set of tree and leaf nodes, respectively
  - $(V_t \cup V_l, E)$ is a directed acyclic graph
  - $E \subseteq V_t \times (V_t \cup V_l)$ is the set of edges
  - $r \in V_t$ is the root node, $\forall x \in V_t \cdot (x, r) \notin E$
- A tree is the set of paths from the root to the leaves
  - assume nodes at each level are enumerated
  - each path is an element of $\mathbb{N}^*$
    - ϵ is the root node
    - $0 \cdot 1 \cdot 0$ means: go to child 0, then 1, then 0

Trees

- A tree τ is a subset of $\mathbb{N}^*$ such that
  - τ is prefix closed
    - $\epsilon \in \tau$
    - $\forall x \in \mathbb{N}^* \cdot \forall y \in \mathbb{N} \cdot (x \cdot y) \in \tau \Rightarrow x \in \tau$
  - τ is child closed
    - $\forall x \in \mathbb{N}^* \cdot \forall y \in \mathbb{N} \cdot (x \cdot y) \in \tau \Rightarrow \forall z \leq y \cdot (x \cdot z) \in \tau$
  - each node $x \in \tau$ is described by the unique path from the root to x
- A degree $d(x)$ of a node x is the number of successors of x
  - $\forall y < d(x) \cdot (x \cdot y) \in \tau \wedge (x \cdot d(x)) \notin \tau$

Trees

- A tree τ is n-ary iff
  - every non-leaf node has degree $d(x)$
    - $\forall x \in \tau \cdot d(x) = n \vee d(x) = 0$
- A tree is leafless if degree of every node > 0
- A D labeled tree is a tuple (τ, L), where
  - τ is a tree
  - $L : \mathbb{N}^* \to D$ is a labeling function
- A string is 1-ary tree
- An infinite string is a leafless 1-ary tree
- A finite word is a Σ-labeled 1-ary tree
- An infinite word is a Σ-labeled 1-ary leafless tree

Tree Automata

- A tree automaton is a tuple $A = (\Sigma, Q, q_0, \delta, \mathcal{F})$, where
  - Σ is a finite alphabet
  - Q is a finite set of states
  - $q_0 \in Q$ is the initial state
  - δ is the transition relation
    - different depending on the type of the automaton
  - $\mathcal{F} \subseteq Q^\omega$ is the acceptance condition
    - can be Büchi, Rabin, Street, Parity, etc.
- For a deterministic n-ary tree automaton $\delta : Q \times \Sigma \to Q^n$, where $\delta(q, a) = (w_0, \ldots, w_{n-1})$ means
  - if A is in state q, and reads node labeled with a, then
    - A splits into n copies
    - copy i is switched to state w_i, and
    - is sent to the ith successor of the tree node

Example

- deterministic automaton accepting all binary $\{a, b\}$-labeled trees that have a b along every branch
- corresponds to AFb

state	$\delta(q, a)$	$\delta(q, b)$
q_0	(q_0, q_0)	(q_1, q_1)
q_1	(q_1, q_1)	(q_1, q_1)

- acceptance is Büchi $\{q_1\}$

Run and Acceptance

- A run of a deterministic tree automaton on a Σ-labeled n-ary tree (T, V) is a $\mathbb{N}^* \times Q$-labeled tree (T, V_r), where
  - $V_r(x) = (x, q)$ indicates that the automaton read letter $V(x)$ while in state q
  - $V_r(\epsilon) = (\epsilon, q_0)$
  - if $V_r(x) = (x, q)$ and $\delta(q, V(x)) = (w_0, \ldots, w_{n-1})$, then
    - $\forall y < n \cdot (x \cdot y) \in T$, and
    - $V_r(x \cdot y) = (x \cdot y, w_y)$
- A run is accepting iff all of its branches satisfy the acceptance condition

Computational Tree of a Tree Automaton

- A computational tree of A on a tree (T, V) is a tree of all runs of A on (T, V)
  - computational tree of a deterministic tree automaton is linear
- Each node of the computational tree is labeled by a set of histories
- A history is a string $(\mathbb{N} \times Q)^*$ describing a run of an automaton over a single branch of the input tree
- A branch β of a computational tree is accepting iff all infinite histories associated with it are accepting
- A tree is accepted iff exists an accepting branch of the computational tree

Non-Deterministic Tree Automata

- For a non-deterministic tree automaton $\delta : Q \times \Sigma \to 2^{Q^n}$, where $\delta(q, a) = \{W_0, \ldots, W_k\}$ means
  - if A is in state q, and reads a node labeled with a
    - pick $W_i \in \delta(q, a)$ and proceed as a deterministic automaton
- A run of a non-deterministic automaton is defined as for the deterministic case
- A computational tree of a non-deterministic tree automaton is branching
  - a tree is accepted iff there exists an accepting branch of the computational tree
  - or equivalently, iff there exists an accepting run

Example

- non-deterministic binary tree automaton that accepts an $\{a, b\}$-labeled tree if at least one branch contains an a
- corresponds to EFa

state	$\delta(q, a)$	$\delta(q, b)$
q_0	(q_1, q_1)	$\{(q_0, q_1), (q_1, q_0)\}$
q_1	(q_1, q_1)	(q_1, q_1)

- acceptance is Büchi $\{q_1\}$

Symbolic Transition Relation

- For deterministic and non-deterministic tree automata transition relation can be described by a boolean formula over $\mathbb{N} \times Q$
- For deterministic binary tree automaton
  - $\delta(q, a) = (w_0, w_1)$ becomes
  - $\delta(q, a) = (0, w_0) \wedge (1, w_1)$
- For a non-deterministic binary tree automaton a choice is encoded by a disjunction
  - $\delta(q, a) = \{(w_0, w_1), (w_2, w_3)\}$ becomes
  - $\delta(q, a) = ((0, w_0) \wedge (1, w_1)) \vee ((0, w_2) \wedge (1, w_3))$
  - note that both conjunction and disjunction are used

Alternating Tree Automata

- For a set X, let $\mathcal{B}(X)$ denote the set of all positive boolean formulas over X
- A set $Y \subseteq X$ satisfies a formula $\theta \in \mathcal{B}(X)$ if treating atoms in Y as true, and in $X \setminus Y$ as false, makes θ true
  - $X = \{a, b, c\}$
  - $\{a, b\}$ satisfies $a \wedge b \vee c$, and
  - does not satisfy $a \wedge b \wedge c$
- An alternating n-ary tree automaton is a tree automaton with transition relation $\delta(q, a) \in \mathcal{B}(\{0, \ldots, n-1\} \times Q)$
  - $(0, q_1) \vee (0, q_2) \wedge (1, q_1)$

Example

- Alternating automaton that accepts all binary $\{a, b\}$-labeled trees where b occurs as a child of every node at the second level
- Corresponds to $AXEXb$

state	$\delta(q, a)$	$\delta(q, b)$
q_0	$(0, q_1) \wedge (1, q_1)$	$(0, q_1) \wedge (1, q_1)$
q_1	$(0, q_2) \vee (1, q_2)$	$(0, q_2) \vee (1, q_2)$
q_2	$(0, q_4) \wedge (1, q_4)$	$(0, q_3) \wedge (1, q_3)$
q_3	$(0, q_3) \wedge (1, q_3)$	$(0, q_3) \wedge (1, q_3)$
q_4	$(0, q_4) \wedge (1, q_4)$	$(0, q_4) \wedge (1, q_4)$

- acceptance is Büchi $\{q_3\}$

Alternating Automata

- A run of an alternating n-ary tree automaton A over a Σ-labeled tree (T, V) is a $\mathbb{N}^* \times Q$ labeled tree (T_r, V_r)
  - $V_r(\epsilon) = (\epsilon, q_0)$
  - if $V_r(x) = (y, q)$ and $\delta(q, a) = \theta$, then there exists a possibly empty set $Y = \{(c_0, w_0), \ldots, (c_k, w_k)\}$ such that
    - Y satisfies θ, and
    - for all $0 \leq i \leq k$, $x \cdot i \in T_r$, and $V_r(x \cdot i) = (y \cdot c_i, w_i)$
- A tree (T, V) is accepted by A iff there exists an accepting run of A over (T, V)

ATA Computational Tree

- As before, we can build a computational tree of A over a Σ-labeled tree (T, V)
- Nodes in the computational tree are labeled with histories
- A tree is accepted by the automaton iff there exists an accepting infinite branch in the computational tree

Example

- Automaton over binary $\{a\}$-labeled tree

state	$\delta(q, a)$
q_0	$(0, q_0) \wedge (1, q_2) \vee (0, q_1)$
q_1	$(0, q_1) \wedge (0, q_2) \wedge (1, q_2)$
q_2	$(0, q_2)$

- computational tree is branching

Extending to Arbitrary Trees

- We only considered trees with a fixed branching degree
- Let $\mathcal{D} \subseteq \mathbb{N}$
  - a $\mathcal{D}$-tree is a tree such that a branching degree of every node is in $\mathcal{D}$
    - $\forall x \cdot d(x) \in \mathcal{D}$
- A $\mathcal{D}$-tree automaton has transition relation $\delta : Q \times \Sigma \times \mathcal{D} \to \mathcal{B}(\mathbb{N} \times Q)$
  - δ is defined separately for each branching degree
  - $\delta(q, a, k)$ can only contain terms from $\{0, k-1\} \times Q$
- A size of a $\mathcal{D}$-tree automaton $A_{\mathcal{D}}$ is
  - $||A_{\mathcal{D}}|| = |\mathcal{D}| + |Q| + |F| + ||\delta||$
  - $||\delta|| = \sum_{q,a,k} |\delta(q, a, k)|$ where $\delta(q, a, k) \neq \mathbf{false}$

Model: Kripke Structure

- As usual, our models are Kripke structures $K = (AP, S, s_0, R, L)$
  - AP is the set of atomic propositions
  - S is a finite set of states
  - $s_0 \in S$ an initial state
  - $R \subseteq S \times S$ the transition relation
  - $L : S \to 2^{AP}$ is the labeling function
- A Kripke structure induces a S-labeled tree (T_K, V_K)
  - $V_K : \mathbb{N}^* \to S$ labels each node with a state
    - $V_K(\epsilon) = s_0$
  - $T_K \subseteq \mathbb{N}^*$ is a tree such that
    - for $y \in T_K$ with $R(V_K(y)) = (w_0, \ldots w_m)$ we have $\forall 0 \leq i \leq m \cdot (y \cdot i) \in T_K$ and $V_K(y \cdot i) = w_i$

Computation Tree

- A Kripke structure can be seen as a computation tree over its atomic propositions
- For a Kripke structure K
  - (T_K, V_K) is its tree unrolling
  - $(T_K, L \circ V_K)$ is its computation tree

Temporal Logic: CTL

- Computation Tree Logic is interpreted over a computation tree of a Kripke structure
- Definition

$$
\begin{array}{lcl}
||p||(s) & = & p \in L(s,p) \\
||\neg\varphi||(s) & = & \neg||\varphi||(s) \\
||\varphi \wedge \psi||(s) & = & ||\varphi||(s) \wedge ||\psi||(s) \\
||\varphi \vee \psi||(s) & = & ||\varphi||(s) \vee ||\psi||(s) \\
||EX\varphi||(s) & = & \exists t \in R(s) \cdot ||\varphi||(t) \\
||AX\varphi||(s) & = & \forall t \in R(s) \cdot ||\varphi||(t) \\
||E[\varphi U \psi]||(s) & = & ||\mu Z \cdot \psi \vee \varphi \wedge EXZ||(s) \\
||A[\varphi U \psi]||(s) & = & ||\mu Z \cdot \psi \vee \varphi \wedge AXZ||(s) \\
||E[\varphi R \psi]||(s) & = & ||\nu Z \cdot \psi \wedge (\varphi \vee EXZ)||(s) \\
||A[\varphi R \psi]||(s) & = & ||\nu Z \cdot \psi \wedge (\varphi \vee AXZ)||(s)
\end{array}
$$

From CTL to ATA

- For a CTL formula φ we construct an alternating $\mathcal{D}$-tree automaton $A_{\mathcal{D},\varphi}$ that accepts all $\mathcal{D}$-trees that are models of φ
- $A_{\mathcal{D},\varphi} = (2^{AP}, cl(\varphi), \varphi, \delta, F)$
  - the alphabet is all subsets of AP
  - states correspond to sub-formulas of φ
  - initial state is φ
  - acceptance condition is Büchi and consists of all AR and ER sub-formulas
  - δ is the transition relation
- Intuitively, $A_{\mathcal{D},\varphi}$ accepts a tree from a state q iff the tree is the model of the formula associated with q

From CTL to ATA

$$
\begin{array}{lcl}
\delta(p,\sigma,k) & = & \textbf{true if } p \in \sigma \\
\delta(p,\sigma,k) & = & \textbf{false if } p \notin \sigma \\
\delta(\neg p,\sigma,k) & = & \textbf{false if } p \in \sigma \\
\delta(\neg p,\sigma,k) & = & \textbf{true if } p \notin \sigma \\
\delta(\varphi \wedge \psi,\sigma,k) & = & \delta(\varphi,\sigma,k) \wedge \delta(\psi,\sigma,k) \\
\delta(\varphi \vee \psi,\sigma,k) & = & \delta(\varphi,\sigma,k) \vee \delta(\psi,\sigma,k) \\
\delta(AX\varphi,\sigma,k) & = & \bigwedge_{c=0}^{k-1}(c,\varphi) \\
\delta(EX\varphi,\sigma,k) & = & \bigvee_{c=0}^{k-1}(c,\varphi) \\
\delta(A[\varphi U \psi],\sigma,k) & = & \delta(\psi,\sigma,k) \vee \delta(\varphi,\sigma,k) \wedge \bigwedge_{c=0}^{k-1}(c, A[\varphi U \psi]) \\
\delta(A[\varphi R \psi],\sigma,k) & = & \delta(\psi,\sigma,k) \wedge (\delta(\varphi,\sigma,k) \vee \bigwedge_{c=0}^{k-1}(c, A[\varphi R \psi])) \\
\delta(E[\varphi U \psi],\sigma,k) & = & \delta(\psi,\sigma,k) \vee \delta(\varphi,\sigma,k) \wedge \bigvee_{c=0}^{k-1}(c, E[\varphi U \psi]) \\
\delta(E[\varphi R \psi],\sigma,k) & = & \delta(\psi,\sigma,k) \wedge (\delta(\varphi,\sigma,k) \vee \bigvee_{c=0}^{k-1}(c, E[\varphi R \psi]))
\end{array}
$$

Examples

- $\psi = AFAGp$
  - in negation normal form: $A[\mathbf{true}\ U\ (A[\mathbf{false}\ R\ p])]$
  - alphabet $2^{\{p\}}$

state	$\delta(q, \{p\}, k)$	$\delta(q, \emptyset, k)$
ψ	$\bigwedge_{c=0}^{k-1}(c, A[\mathbf{false}\ R\ p]) \vee \bigwedge_{c=0}^{k-1}(c, \psi)$	$\bigwedge_{c=0}^{k-1}(c, \psi)$
$A[\mathbf{false}\ R\ p]$	$\bigwedge_{c=0}^{k-1}(c, A[\mathbf{false}\ R\ p])$	$\mathbf{false}$

- acceptance condition is Büchi $\{A[\mathbf{false}\ R\ p]\}$

Examples

- $\psi = A[(\neg AXp)\ U\ b]$
  - in negation normal form: $A[(EX\neg p)\ U\ q]$
  - alphabet $2^{\{p,b\}}$

state	$\delta(q,\{p,b\},k)$	$\delta(q,\{p\},k)$	$\delta(q,\{b\},k)$	$\delta(q,\emptyset,k)$
ψ	**true**	$\bigvee_{c=0}^{k-1}(c,\neg p) \wedge \bigwedge_{c=0}^{k-1}(c,\psi)$	**true**	$\bigvee_{c=0}^{k-1}(c,\neg p) \wedge \bigwedge_{c=0}^{k-1}(c,\psi)$
$\neg p$	**false**	**false**	**true**	**true**