Web-Mining Agents
Data Mining

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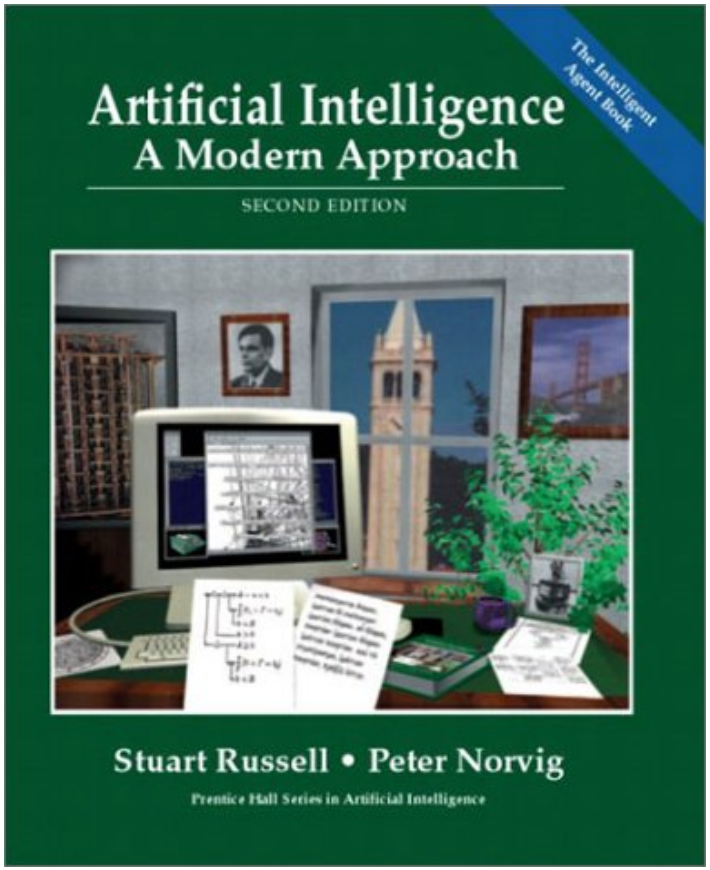
Literature



- Chapter 14 (Section 1 and 2)

Syntax and Semantics of Bayesian Networks

Bayesian networks

- A simple, graphical notation for conditional independence assertions and hence for compact specification of full joint distributions

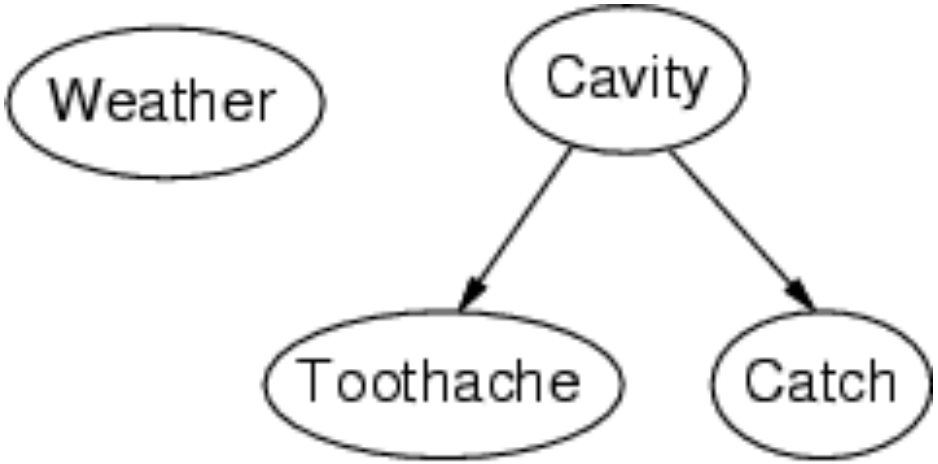
- Syntax:
  - a set of nodes, one per variable
  - a directed, acyclic graph (link ≈ "directly influences")
  - a conditional distribution for each node given its parents:

$$\mathbf{P}\,(X_i \mid \text{Parents}\,(X_i))$$

- In the simplest case, conditional distribution represented as a conditional probability table (CPT) giving the distribution over X_i for each combination of parent values

Example

- Topology of network encodes conditional independence assertions:



- Weather is independent of the other variables
- Toothache and Catch are conditionally independent given Cavity

Example

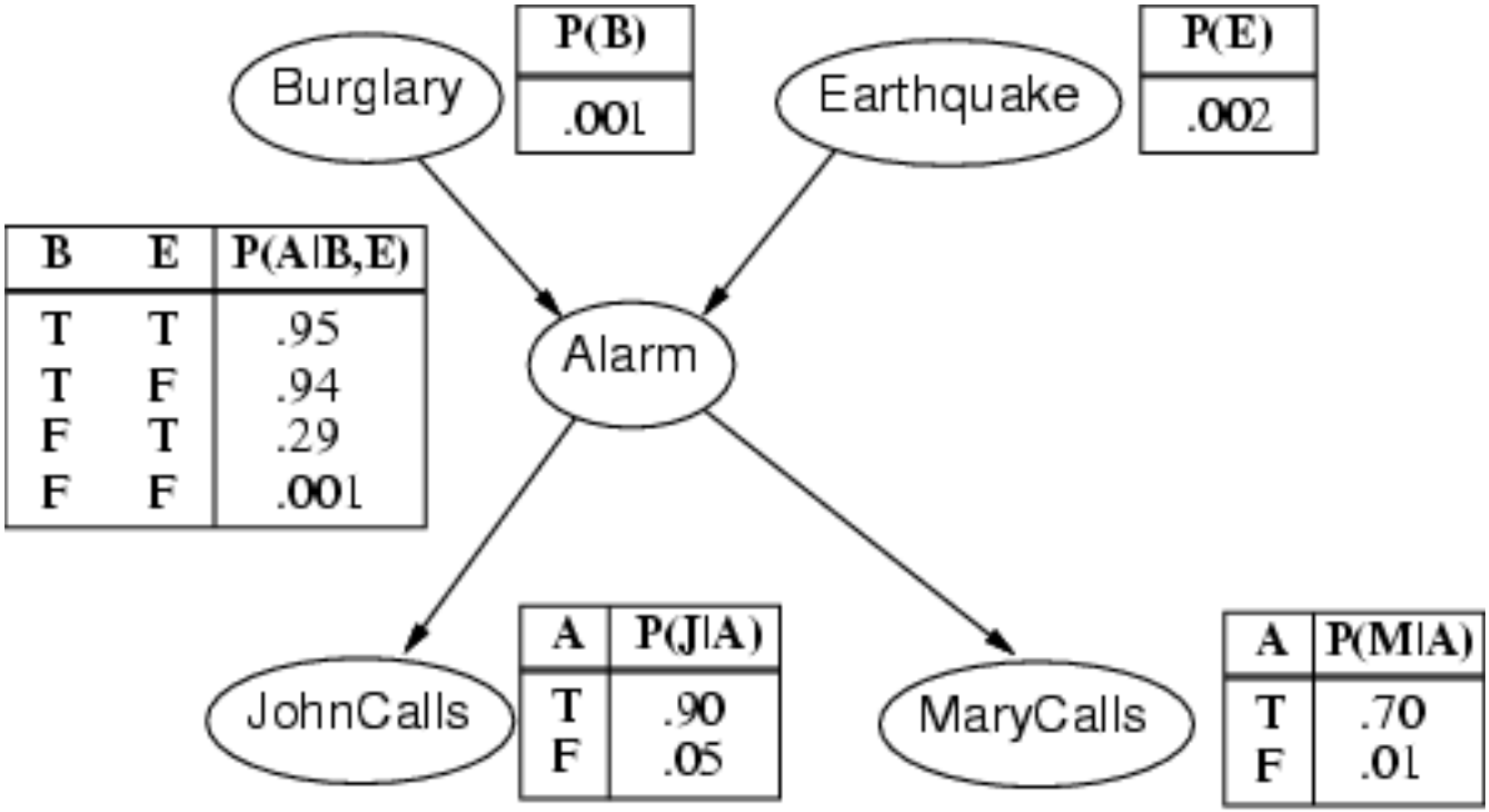
- I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

- Variables: Burglary, Earthquake, Alarm, JohnCalls, MaryCalls

- In this case, the network topology reflects "causal" knowledge:
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm can cause Mary to call
  - The alarm can cause John to call

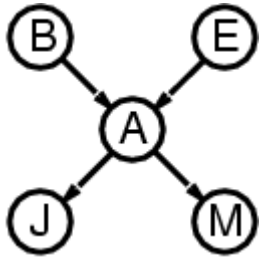
A formal in-depth treatment of causality will be given in lectures 5 to 9

Example contd.



Compactness

- A CPT for Boolean X_i with k Boolean parents has 2^k rows for the combinations of parent values

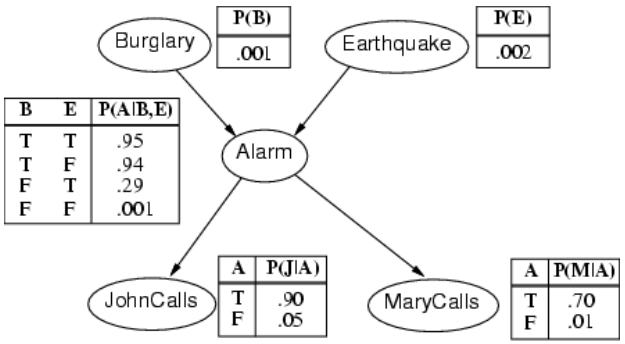


- Each row requires one number p for X_i = true (the number for X_i = false is just 1-p)
- If each variable has no more than k parents, the complete network requires $O(n \cdot 2^k)$ numbers
- i.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution
- For burglary net, 1 + 1 + 4 + 2 + 2 = 10 numbers (vs. 2^5-1 = 31)

Semantics

The full joint distribution of a BN is defined as the product of the local conditional distributions:

$$\mathbf{P}(X_1, \ldots, X_n) = \Pi_{i=1}^{n} \mathbf{P}(X_i \mid \text{Parents}(X_i))$$



e.g., $P(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

$= P(j \mid a)\, P(m \mid a)\, P(a \mid \neg b, \neg e)\, P(\neg b)\, P(\neg e)$

$= 0.90 \times 0.7 \times 0.001 \times 0.999 \times 0.998$

≈ 0.00063

Constructing Bayesian networks

- 1. Choose an ordering of variables $X_1, \ldots, X_n$
- 2. For $i = 1$ to n
  - add X_i to the network
  - select parents from $X_1, \ldots, X_{i-1}$ such that

$$\mathbf{P}(X_i \mid \text{Parents}(X_i)) = \mathbf{P}(X_i \mid X_1, \ldots X_{i-1})$$

This choice of parents guarantees:

$$\mathbf{P}(X_1, \ldots, X_n) = \prod_{i=1}^{n} \mathbf{P}(X_i \mid X_1, \ldots, X_{i-1}) \quad \text{(chain rule)}$$

$$= \prod_{i=1}^{n} \mathbf{P}(X_i \mid \text{Parents}(X_i)) \quad \text{(by construction)}$$

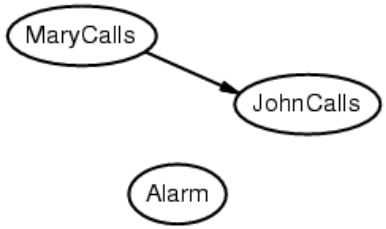
Example

- Suppose we choose the ordering M, J, A, B, E

$\boldsymbol{P}(J \mid M) = \boldsymbol{P}(J)?$

Example

- Suppose we choose the ordering M, J, A, B, E

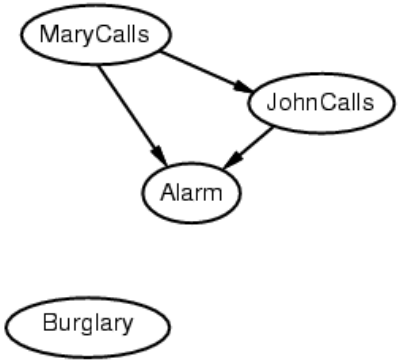


P(J | M) = **P**(J)? **No**

P(A | J, M) = **P**(A)? **P**(A | J, M) = **P**(A | J)?

Example

- Suppose we choose the ordering M, J, A, B, E



P(J | M) = **P**(J)? **No**

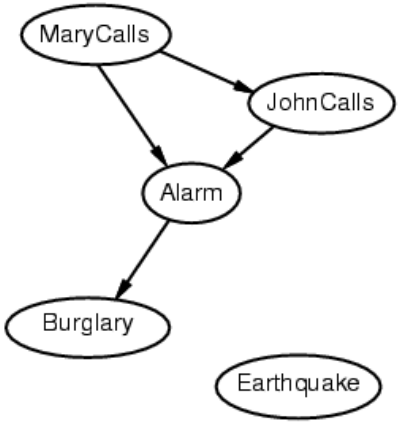
P(A | J, M) = **P**(A)? **P**(A | J, M) = **P**(A | J)? **No**

P(B | A, J, M) = **P**(B | A)?

P(B | A, J, M) = **P**(B)?

Example

- Suppose we choose the ordering M, J, A, B, E



P(J | M) = **P**(J)? **No**

P(A | J, M) = **P**(A | J)? **P**(A | J, M) = **P**(A)? **No**

P(B | A, J, M) = **P**(B | A)? **Yes**

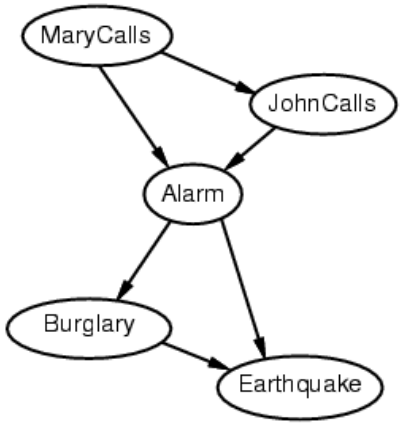
P(B | A, J, M) = **P**(B)? **No**

P(E | B, A ,J, M) = **P**(E | A)?

P(E | B, A, J, M) = **P**(E | A, B)?

Example

- Suppose we choose the ordering M, J, A, B, E



P(J | M) = **P**(J)? **No**

P(A | J, M) = **P**(A | J)? **P**(A | J, M) = **P**(A)? **No**

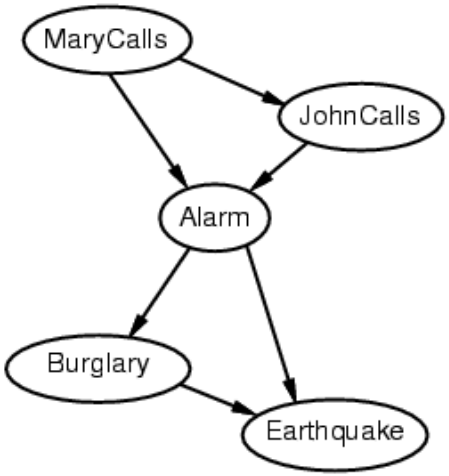
P(B | A, J, M) = **P**(B | A)? **Yes**

P(B | A, J, M) = **P**(B)? **No**

P(E | B, A ,J, M) = **P**(E | A)? **No**

P(E | B, A, J, M) = **P**(E | A, B)? **Yes**

Example contd.

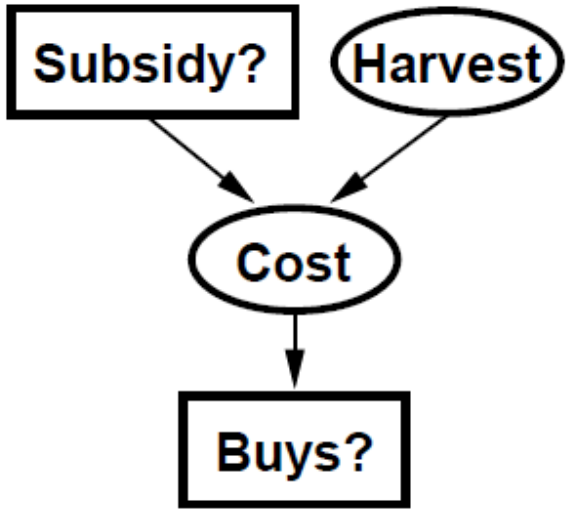


- Deciding conditional independence is hard in non-causal directions
- (Causal models and conditional independence seem hardwired for humans!)
- Network is less compact: 1 + 2 + 4 + 2 + 4 = 13 numbers needed instead of 10.

Hybrid (discrete+continuous) networks

BNs can also deal with continuous RVs or even hybrid ones

Discrete (*Subsidy?* and *Buys?*); continuous (*Harvest* and *Cost*)



Option 1: discretization—possibly large errors, large CPTs
Option 2: finitely parameterized canonical families

1) Continuous variable, discrete+continuous parents (e.g., *Cost*)
2) Discrete variable, continuous parents (e.g., *Buys?*)

Exact Inference on Bayesian Networks

Inference tasks

Simple queries: compute posterior marginal $\mathbf{P}(X_i|\mathbf{E}=\mathbf{e})$
e.g., $P(NoGas|Gauge=empty, Lights=on, Starts=false)$

Conjunctive queries: $\mathbf{P}(X_i, X_j|\mathbf{E}=\mathbf{e}) = \mathbf{P}(X_i|\mathbf{E}=\mathbf{e})\mathbf{P}(X_j|X_i, \mathbf{E}=\mathbf{e})$

Optimal decisions: decision networks include utility information;
probabilistic inference required for $P(outcome|action, evidence)$

Value of information: which evidence to seek next?

Sensitivity analysis: which probability values are most critical?

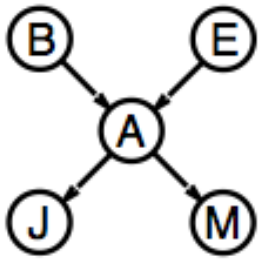
Explanation: why do I need a new starter motor?

Inference by enumeration

Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation

Simple query on the burglary network:

$$\begin{aligned}&\mathbf{P}(B|j,m)\\&= \mathbf{P}(B,j,m)/P(j,m)\\&= \alpha\mathbf{P}(B,j,m)\\&= \alpha \textstyle\sum_e \sum_a \mathbf{P}(B,e,a,j,m)\end{aligned}$$



Rewrite full joint entries using product of CPT entries:

$$\begin{aligned}&\mathbf{P}(B|j,m)\\&= \alpha \textstyle\sum_e \sum_a \mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a)\\&= \alpha\mathbf{P}(B) \textstyle\sum_e P(e) \sum_a \mathbf{P}(a|B,e)P(j|a)P(m|a)\end{aligned}$$

Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

(Calculation over tree sum-product evaluation tree with path length n (=number of RVs) and degree d (=maximal number domain elements))

Enumeration algorithm

function ENUMERATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayesian network with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$

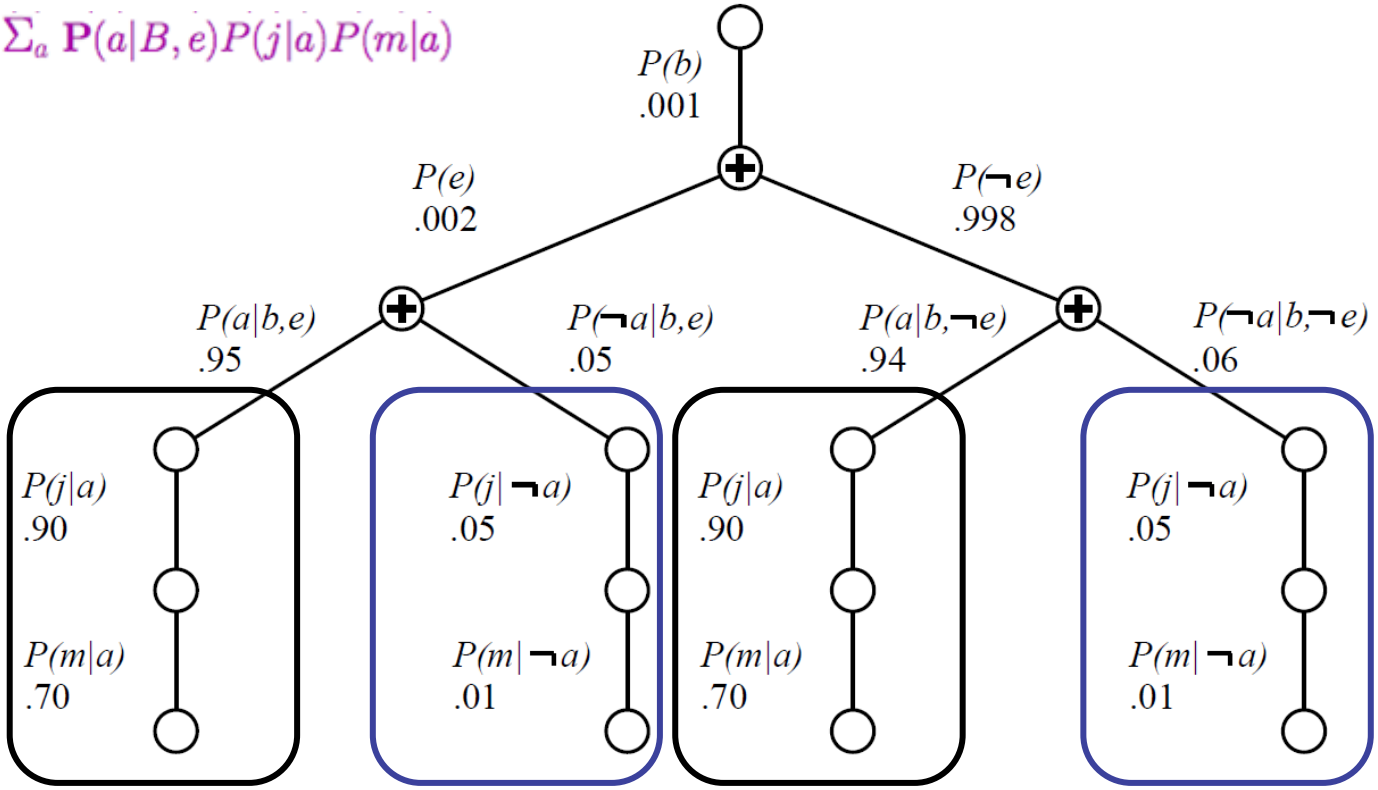
  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    extend $\mathbf{e}$ with value x_i for X
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(VARS[bn], $\mathbf{e}$)
  return NORMALIZE($\mathbf{Q}(X)$)

---

function ENUMERATE-ALL($vars$, $\mathbf{e}$) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $Y \leftarrow$ FIRST($vars$)
  if Y has value y in $\mathbf{e}$
    then return $P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}$)
    else return $\Sigma_y \; P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_y$)
      where $\mathbf{e}_y$ is $\mathbf{e}$ extended with $Y = y$

Evaluation Tree

$\mathbf{P}(B)\ \Sigma_e\ P(e)\ \Sigma_a\ \mathbf{P}(a|B,e)P(j|a)P(m|a)$



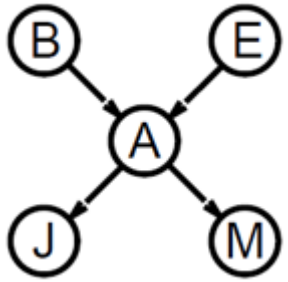
Enumeration is inefficient: repeated computation
e.g., computes $P(j|a)P(m|a)$ for each value of e

Can do better with Variable Elimination (VE)

Basic Objects of VE

$$\mathbf{P}(B) \sum_e P(e) \sum_a \mathbf{P}(a|B,e) P(j|a) P(m|a)$$

- Track objects called factors (right-to-left, in tree: bottom up)
- Initial factors are local CPTs



$$\underbrace{P(B)}_{f_B(B)} \quad \underbrace{P(J|A)}_{f_J(A,J)} \quad \underbrace{P(A|B,E)}_{f_A(A,B,E)}$$

P(m|A)

$f_M(A)$

- During elimination create new factors
- Form of Dynamic Programming

Basic Operations: Pointwise Product

- Pointwise product of factors $\mathbf{f}_1$ and $\mathbf{f}_2$
  - for example: $\mathbf{f}_1(A,B) * \mathbf{f}_2(B,C) = \mathbf{f}(A,B,C)$
  - in general:
    $\mathbf{f}_1(X_1,\ldots,X_j,Y_1,\ldots,Y_k) * \mathbf{f}_2(Y_1,\ldots,Y_k,Z_1,\ldots,Z_l) = \mathbf{f}(X_1,\ldots,X_j,Y_1,\ldots,Y_k,Z_1,\ldots,Z_l)$
  - has 2^{j+k+l} entries (if all variables are binary)

Join by pointwise product

A	$f_{JM}(A)$
T	.9 * .7
F	.05 * .01

=

A	$f_J(A)$
T	.9
F	.05

A	$f_M(A)$
T	.7
F	.01

$$\mathbf{P}(B) \sum_e P(e) \sum_a \mathbf{P}(a|B,e) P(j|a) P(m|a)$$

A	B	E	$f_{AJM}(A,B,E)$
T	T	T	.95 * .63
T	T	F	.94 * .63
T	F	T	.29 * .63
T	F	F	.001 * .63
F	T	T	.05 * .0005
F	T	F	.06 * .0005
F	F	T	.71 * .0005
F	F	F	.999 * .0005

=

A	B	E	$f_A \cdots(A,B,E)$
T	T	T	.95
T	T	F	.94
T	F	T	.29
T	F	F	.001
F	T	T	.05
F	T	F	.06
F	F	T	.71
F	F	F	.999

A	$f_{JM}(A)$
T	.63
F	.0005

Basic Operations: Summing out

- Summing out a variable from a product of factors
  - Move any constant factors outside the summation
  - Add up submatrices in pointwise product of remaining factors

$$\Sigma_x f_1 * \ldots * f_k = f_1 * \ldots * f_i * \Sigma_x f_{i+1} * \ldots * f_k$$
$$= f_1 * \ldots * f_i * f_X$$

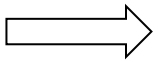
assuming $\mathbf{f}_1, \ldots, \mathbf{f}_i$ do not depend on X

Summing out

$$\mathbf{P}(B) \sum_e P(e) \sum_a \mathbf{P}(a|B,e) P(j|a) P(m|a)$$

A	B	E	$f_{AJM}(A,B,E)$
T	T	T	.95 * .63
T	T	F	.94 * .63
T	F	T	.29 * .63
T	F	F	.001 * .63
F	T	T	.05 * .0005
F	T	F	.06 * .0005
F	F	T	.71 * .0005
F	F	F	.999 * .0005

Summing out a



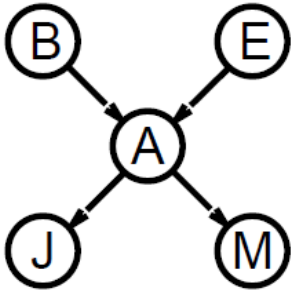
B	E	$f_{\bar{A}JM}(B,E)$
T	T	.95 * .63 + .05 * .0005 = .5985
T	F	.94 * .63 + .06 * .0005 = .5922
F	T	.29 * .63 + .71 * .0005 = .1830
F	F	.001 * .63 + .999 * .0005 = .001129

VE result for Burglary Example

$$
\begin{aligned}
&\mathbf{P}(B|j,m) \\
&\quad = \alpha \underbrace{\mathbf{P}(B)}_{B} \Sigma_e \underbrace{P(e)}_{E} \Sigma_a \underbrace{\mathbf{P}(a|B,e)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M} \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a \mathbf{P}(a|B,e) P(j|a) f_M(a) \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a \mathbf{P}(a|B,e) f_J(a) f_M(a) \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a f_A(a,b,e) f_J(a) f_M(a) \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) f_{\bar{A}JM}(b,e) \text{ (sum out } A\text{)} \\
&\quad = \alpha \mathbf{P}(B) f_{\bar{E}\bar{A}JM}(b) \text{ (sum out } E\text{)} \\
&\quad = \alpha f_B(b) \times f_{\bar{E}\bar{A}JM}(b)
\end{aligned}
$$

Optimization by Finding Irrelevant Variables

Consider the query $P(JohnCalls|Burglary = true)$



$$P(J|b) = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e) P(J|a) \sum_m P(m|a)$$

Sum over m is identically 1; M is **irrelevant** to the query

Thm 1: Y is irrelevant unless $Y \in Ancestors(\{X\} \cup \mathbf{E})$

Here, $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$, and
$Ancestors(\{X\} \cup \mathbf{E}) =$ {Alarm, Earthquake, Burglary}
so $MaryCalls$ is irrelevant

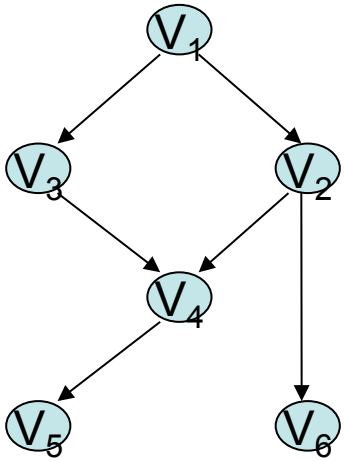
Irrelevant variables can be identified by a graph theoretical criterion on the associated moral graph by m-separation

What else can we do?

- General methodology: Consider special cases (special data structures)
- Here: Consider polytrees (singly connected networks)
- Following slides on Pearl‘s belief propagation algorithm based on slides by Tomas Singliar (adapted by Daniel Lowd)

Polytree Bayesian Networks

- Bayesian network graph is a *polytree (or singly connected network)* iff there is at most one path between any two nodes, V_i and V_k (iff the underlying non-directed graph is a tree)
- Exact BN inference is NP-hard but O(n) on polytree
- (Non-)Examples:



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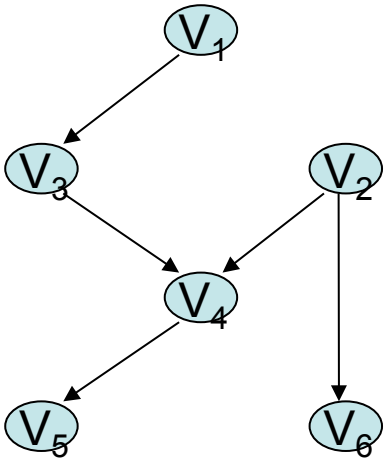

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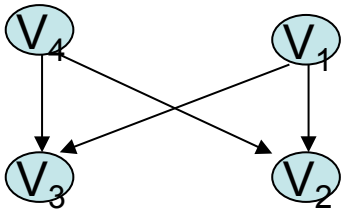

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Our inference task

- We know the values of some evidence variables E:

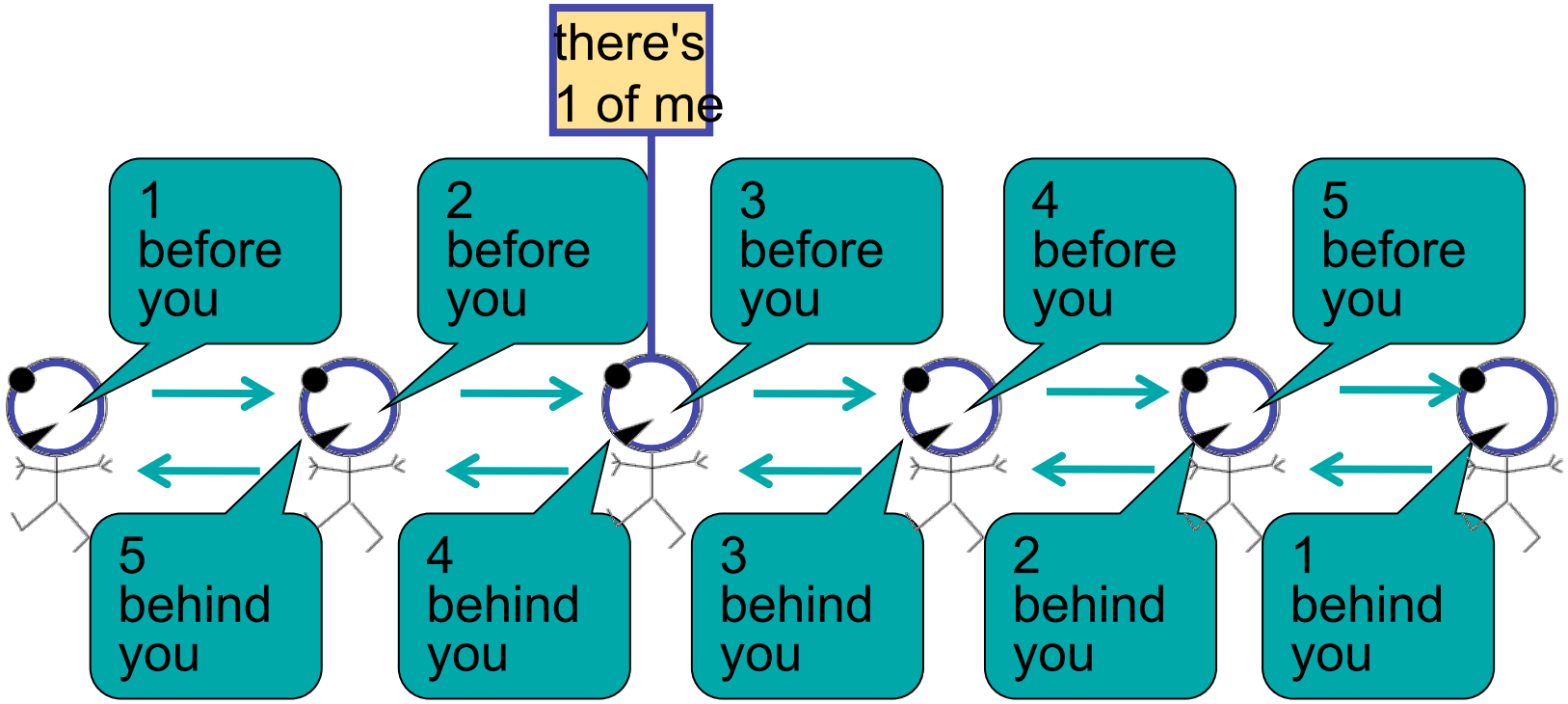
$$V_{e_1},\ldots,V_{e_{|E|}}$$

- We wish to compute the conditional probability $P(X_i \mid E)$ for all non-evidence variables X_i.

- IDEA:
  - Do variable elimination in parallel (locally on each node/ RV)
  - Send/receive required knowledge to/from other nodes

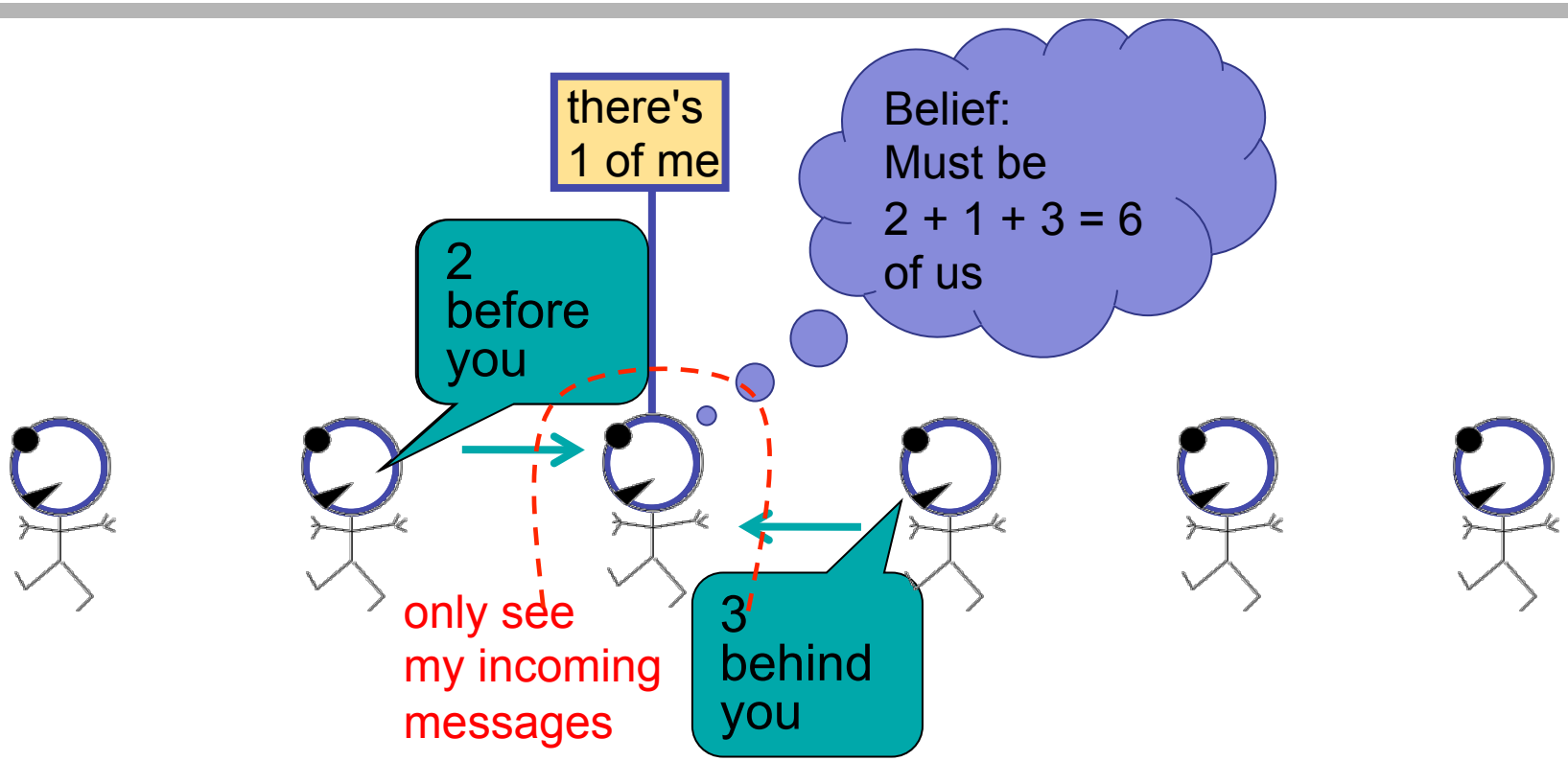
- Following soldier counting example on message passing adapted from slides of M. Gormley

Counting soldiers example (linear)

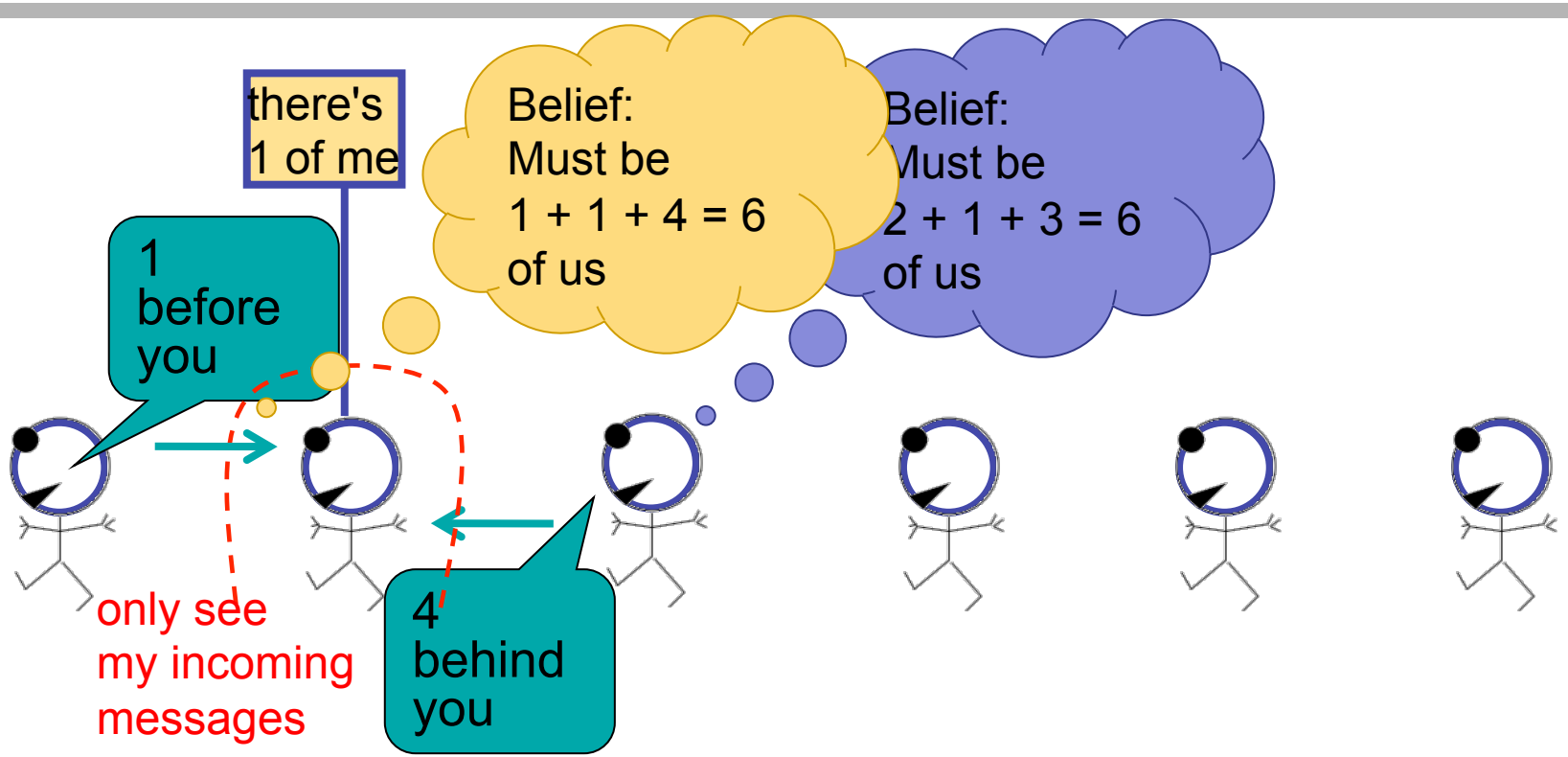


adapted from MacKay (2003): Information Theory, Inference and Learning Algorithms.

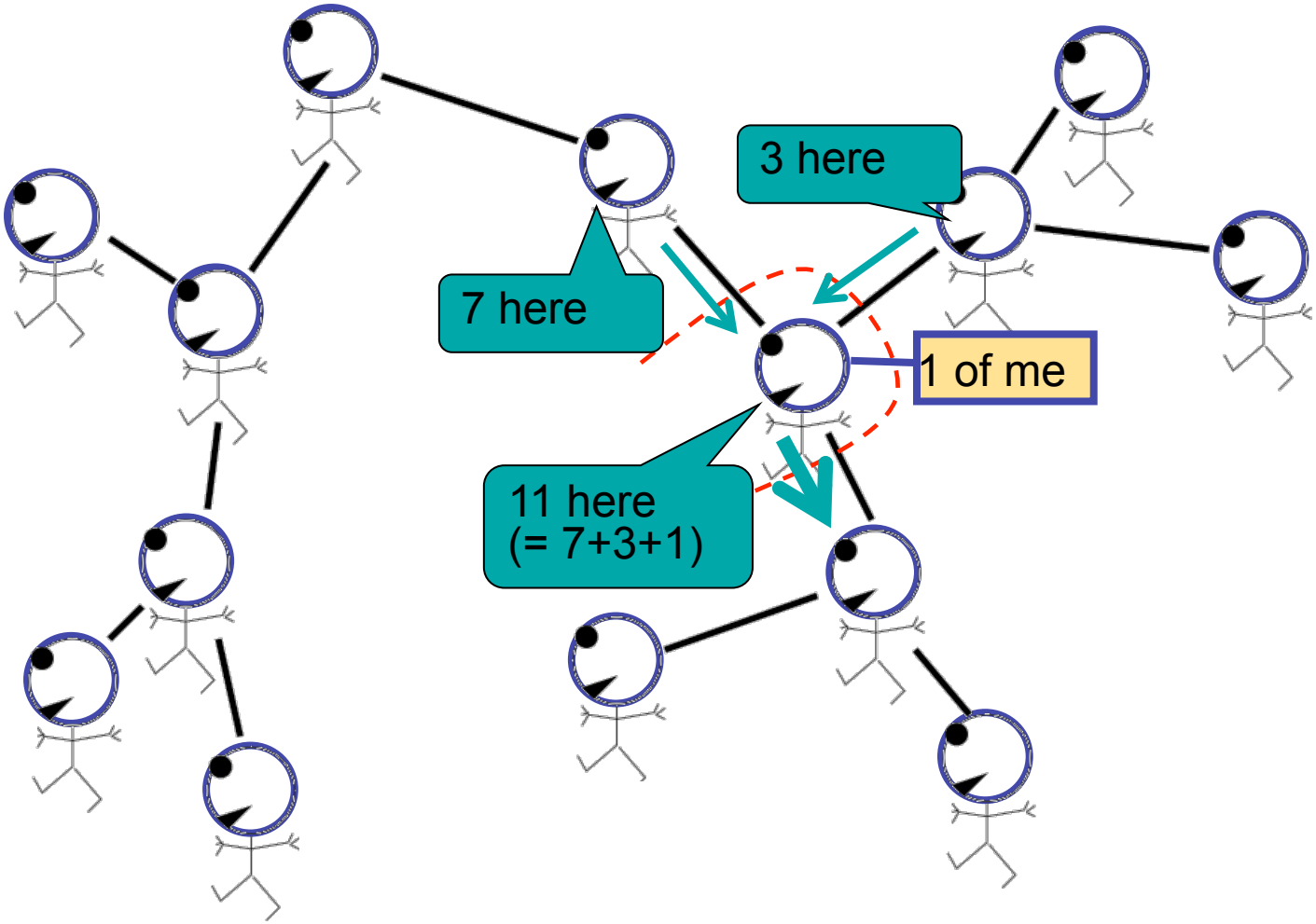
Counting soldiers example (beliefs)



Counting soldiers example (locality)



Counting soldiers example (in polytree)



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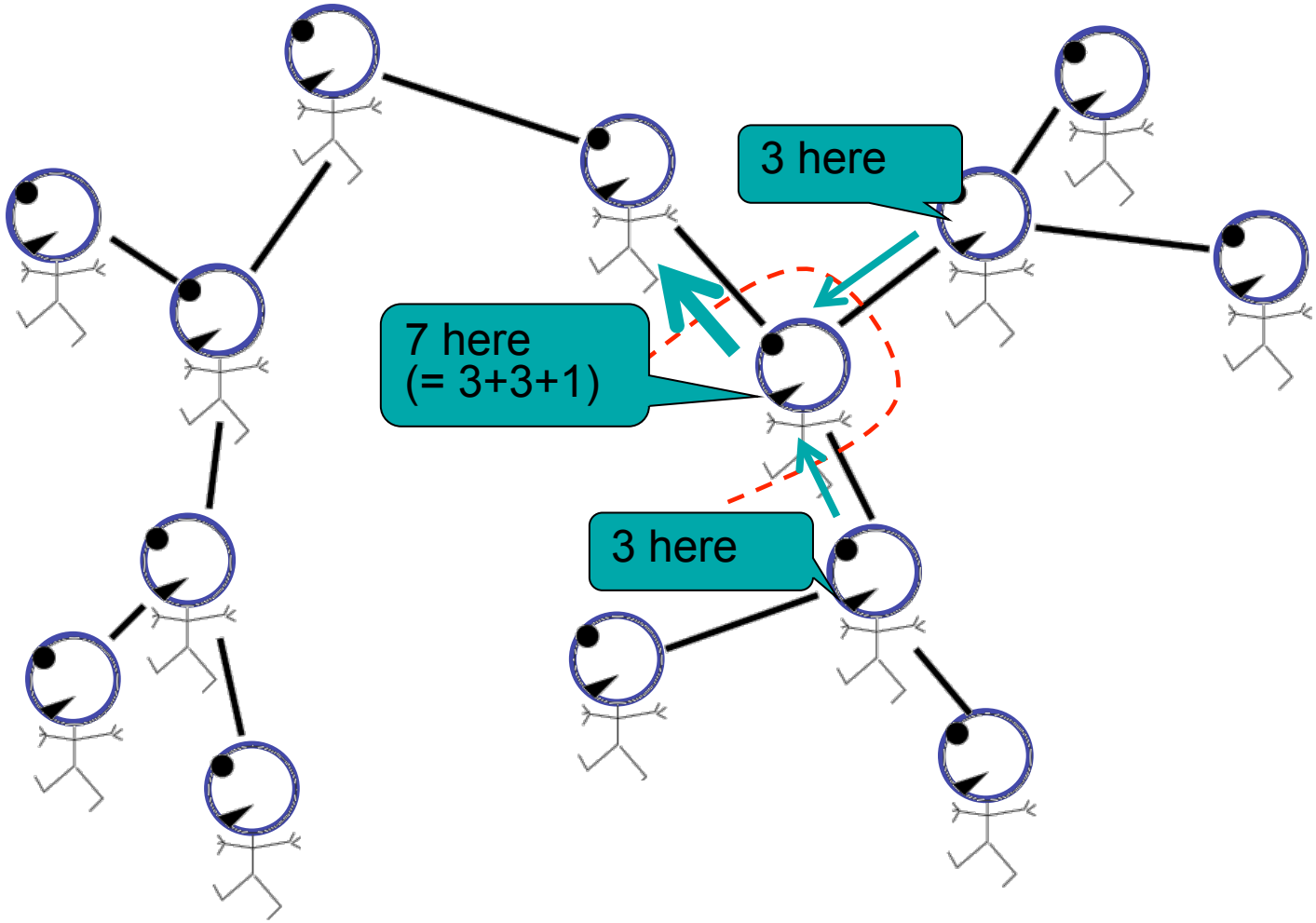

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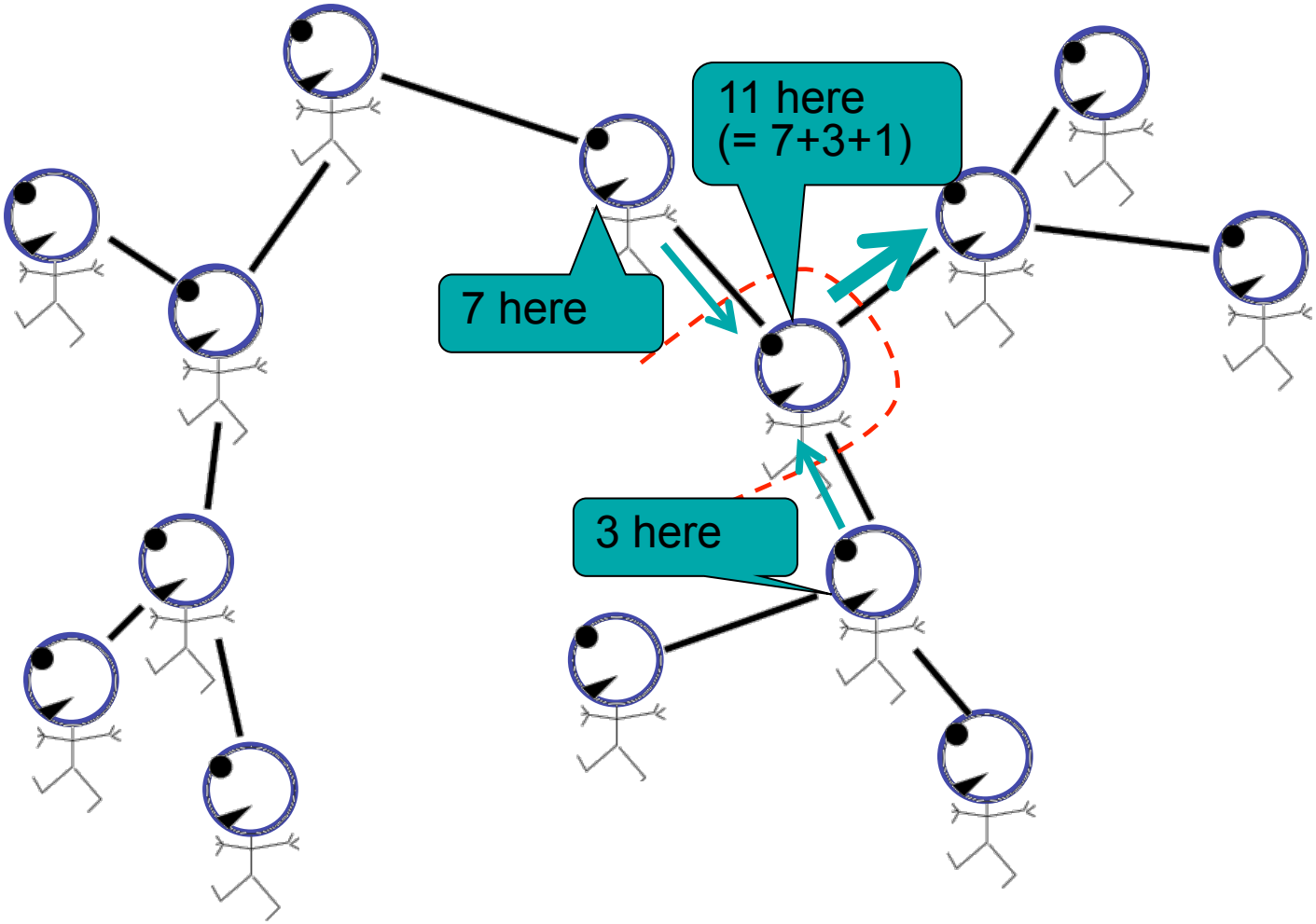
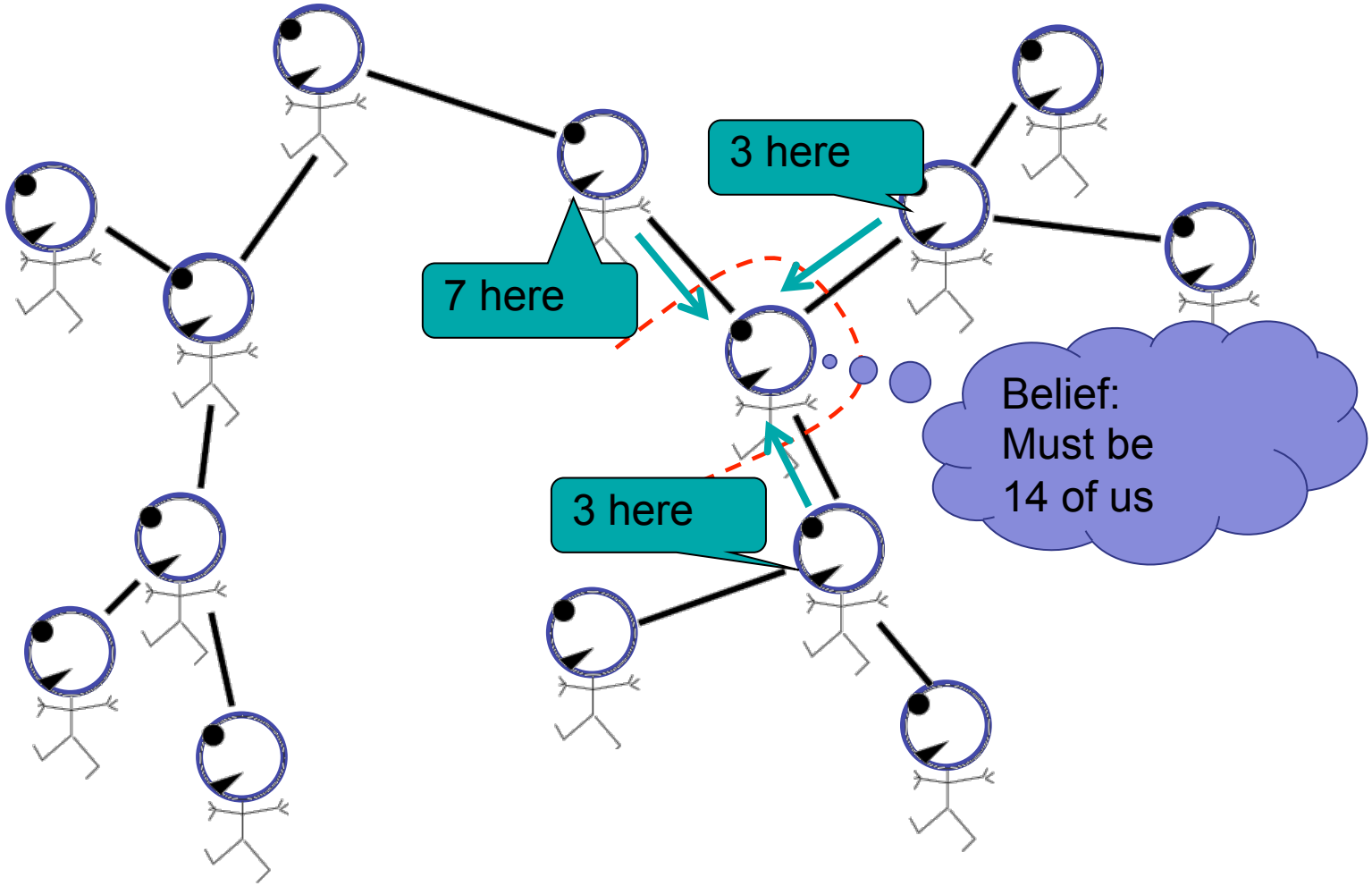
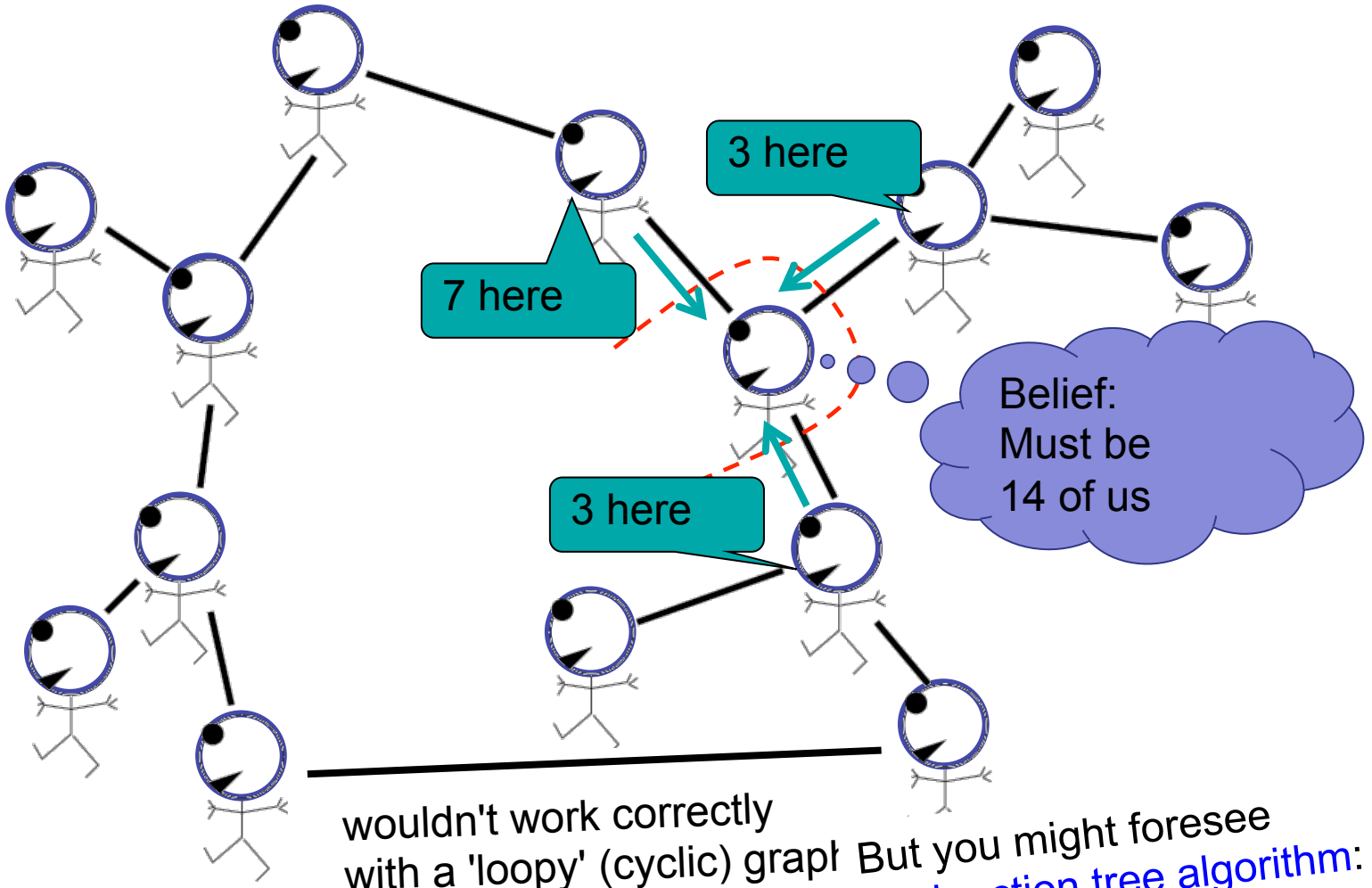


Counting soldiers example (in polytree)



Counting soldiers example (non-circularity)



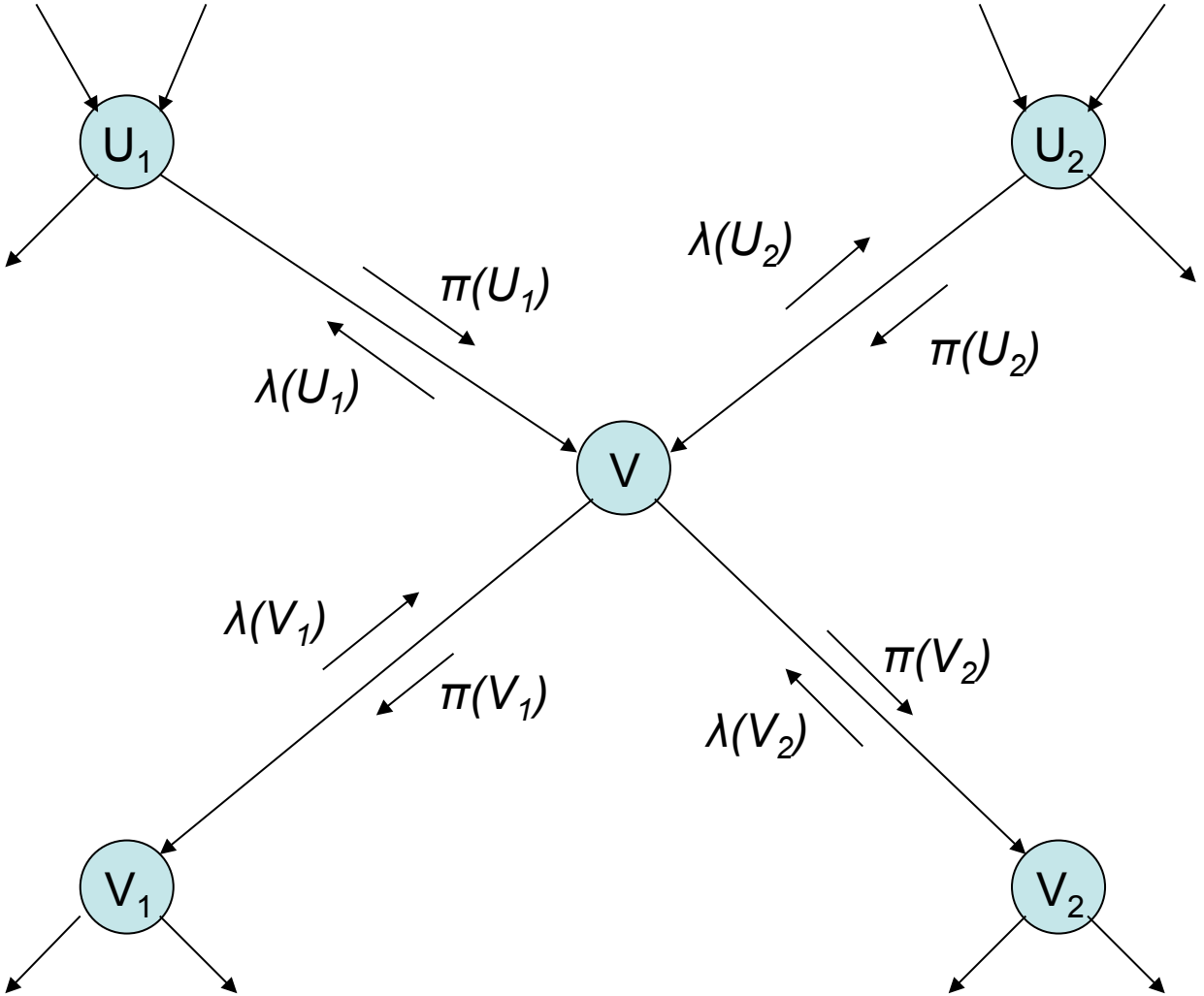
wouldn't work correctly
with a 'loopy' (cyclic) graph

But you might foresee
the junction tree algorithm:
One has to partition the graph
into a tree of clusters

Pearl‘s belief propagation algorithm

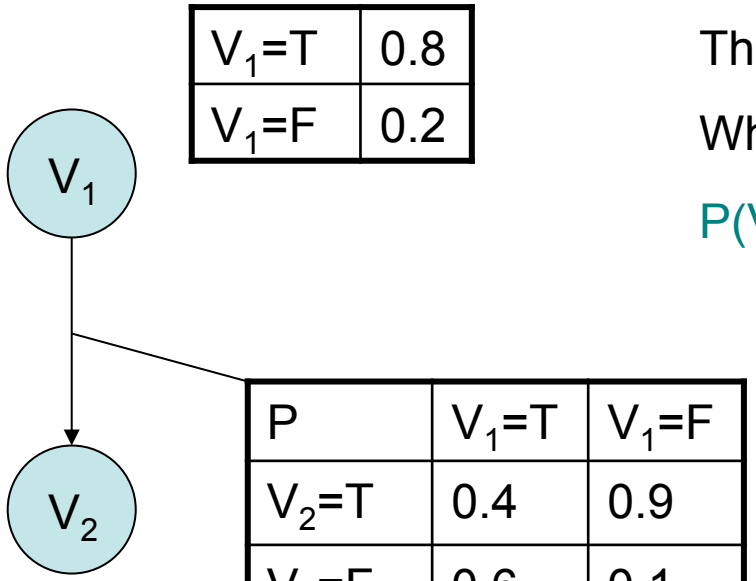
- Local computation for one node V desired
- Information flows through the links of G
  - flows as messages of two types – λ and π
- V splits network into two disjoint parts
  - Strong independence assumptions induced – crucial!
- Denote E_V^+ the part of evidence accessible through the parents of V (causal or predictive)
  - passed downward in π messages
- Analogously, let E_V^- be the diagnostic evidence
  - passed upwards in λ messages

Pearl's Belief Propagation



The π Messages

- What are the messages?
- For simplicity, let the nodes be binary



V_1=T	0.8
V_1=F	0.2

P	V_1=T	V_1=F
V_2=T	0.4	0.9
V_2=F	0.6	0.1

The message passes on information.

What information? Observe:

$P(V_2) = P(V_2|\ V_1=T)P(V_1=T)$

$+ P(V_2|\ V_1=F)P(V_1=F)$

The information needed is the CPT of $V_1 = \pi_V(V_1)$

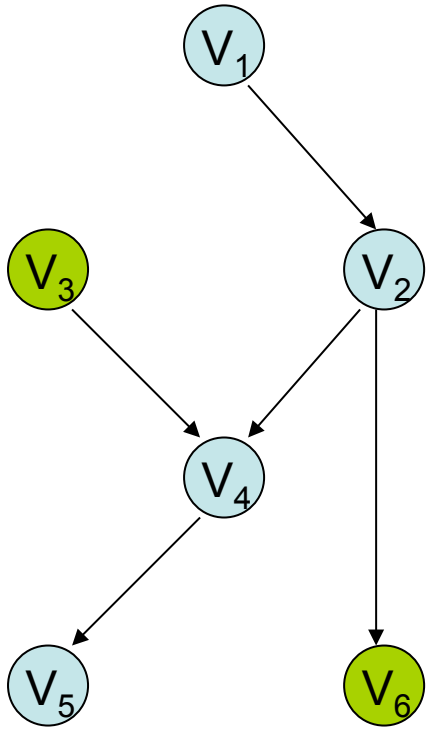
π Messages capture information passed from parent to child

The Evidence

- Evidence – values of observed nodes
  - $V_3 = T$, $V_6 = 3$
- Our belief in what the value of V_i ‘should’ be changes.
- This belief is propagated
- As if the CPTs became

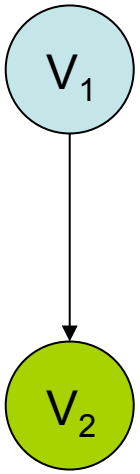
V_3=T	1.0
V_3=F	0.0

P	V_2=T	V_2=F
V_6=1	0.0	0.0
V_6=2	0.0	0.0
V_6=3	1.0	1.0



The Messages

- We know what the π messages are
- What about λ?



Assume E = { V_2 } and compute by Bayes rule:

$$P(V_1 \mid V_2) = \frac{P(V_1)P(V_2 \mid V_1)}{P(V_2)} = \alpha P(V_1)P(V_2 \mid V_1)$$

The information not available at V_1 is $P(V_2|V_1)$. To be passed upwards by a λ-message. Again, this is not in general exactly the CPT, but the belief based on evidence down the tree.

- The messages are $\pi(V)=P(V|E^+)$ and $\lambda(V)=P(E^-|V)$

Combination of evidence in node V

- Recall that $E_V = E_V^+ \cup E_V^-$ and let us compute

$$P(V \mid E) = P(V \mid E_V^+, E_V^-) = \alpha' P(E_V^+, E_V^- \mid V) P(V) =$$
$$\alpha' P(E_V^- \mid V) P(E_V^+ \mid V) P(V) = \alpha P(E_V^- \mid V) P(V \mid E_V^+) =$$
$$\alpha \lambda(V) \pi(V) = BEL(V)$$

- α is the normalization constant ($1/P(E^-|E^+)$)
- normalization is not necessary (can do it at the end)
- but may prevent numerical underflow problems

λ message combination

- Assume X received λ-messages from neighbors
- How to compute $\lambda(x) = P(e^-|x)$?
- Let $Y_1, \ldots, Y_c$ be the children of X
- $\lambda_{XY}(x)$ denotes the λ-message sent between X and Y

$$\lambda(x) = \prod_{j=1}^{c} \lambda_{Y_j X_i}(x)$$

Derivation clear: Assume two children Y, Z of X, then $P(e^-|x) = P(e^-_Y, e^-_Z|x) = P(e^-_Y|x) * P(e^-_Z|x)$ (siblings independent given parent)

π message combination

- Assume X received π -messages from neighbors
- How to compute $\pi(x) = P(x|e^+)$?
- Let $U_1, \ldots, U_p$ be the parents of X
- $\pi_{XY}(x)$ denotes the π-message sent between X and Y
- summation over the CPT

$$\pi(x) = \sum_{u_1,\ldots,u_p} P(x \mid u_1,\ldots,u_p) \prod_{j=1}^{p} \pi_{U_j X_i}(u_j)$$

Derivation: Condition on all U_i and note that each pair (parent U_i,evidence e^+_i) is independent of the other pairs (U_j,e^+_j)

Messages to pass

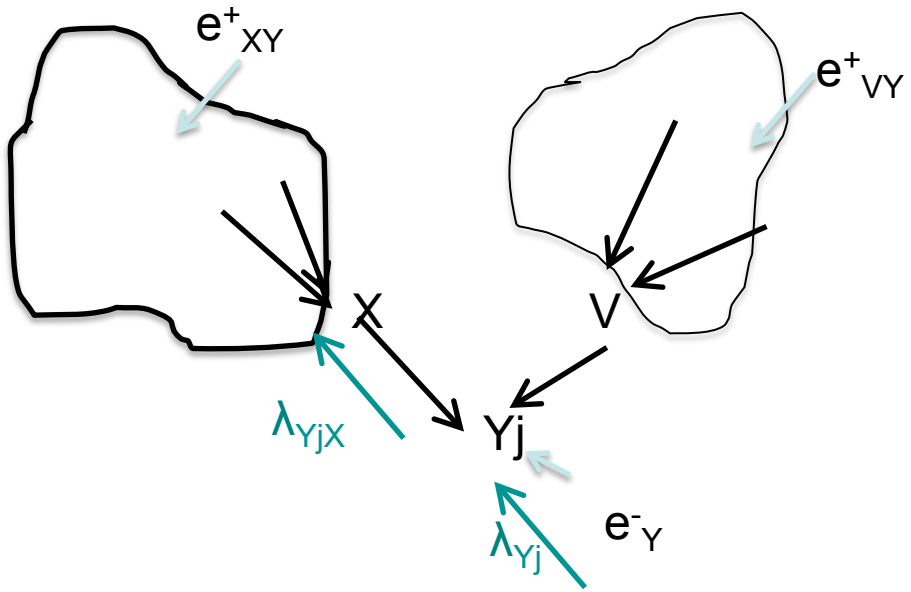
- We need to compute $\pi_{XY}(x)$

$$\pi_{XY_J}(x) = \alpha \pi_X(x) \prod_{k \neq j} \lambda_{Y_k X}(x)$$

Derivation

π_{XYJ}
$= P(x \mid e^{+}_{XYj})$
$= P(x \mid e - e^{-}_{XYj}) =$
= [Bel(x) when evidence e^{-}_{Xyj} is surpressed]
= Bel(x) setting $\lambda_{XYj}(x) = 1$
= formula above

Messages to pass



Derivation hint
- $\lambda_{YjX} = P(e^-_{XYj}|x)$
  $= P(e^+_{VYj}, e^-_{Yj}|X)$
- then condition on YJ and V,
- And use independences
  - Y_j seperates e^+_{VYj} from e^-_{Yj}
  - V seperates e^+_{VYj} from X

- Symbolically, group other (≠X) parents of Yj into single complex $V = V_1, \ldots, V_q$ and treat link X->Yj vs. V-> Yj

$$\lambda_{Y_jX}(x) = \sum_{y_j} \lambda_{Y_j}(y_j) \sum_{v_1,\ldots,v_q} p(y \mid v_1,\ldots,v_q) \prod_{k=1}^{q} \pi_{V_kY_j}(v_k)$$

Pearl's Belief Propagation Algorithm

- Initialization step
  - For all $V_i=e_i$ in E:
    - $\lambda(x_i) = 1$ whenever $x_i = e_i$; 0 otherwise
    - $\pi(x_i) = 1$ whenever $x_i = e_i$; 0 otherwise
  - For nodes without parents
    - $\pi(x_i) = p(x_i)$ - prior probabilities
  - For nodes without children
    - $\lambda(x_i) = 1$ uniformly (normalize at end)

Pearl's Belief Propagation Algorithm

- Iterate until no change occurs
  - (For each node X) if X has received all the π messages from its parents, calculate π(x)
  - (For each node X) if X has received all the λ messages from its children, calculate λ(x)
  - (For each node X) if π(x) has been calculated and X received all the λ-messages from all its children (except Y), calculate $\pi_{XY}(x)$ and send it to Y.
  - (For each node X) if λ(x) has been calculated and X received all the π-messages from all parents (except U), calculate $\lambda_{XU}(u)$ and send it to U.
- Compute BEL(X) = λ(x)π(x) and normalize

Complexity

- On a polytree, the BP algorithm converges in time proportional to diameter of network – at most linear
- Work done in a node is proportional to the size of CPT
- Hence BP is linear in number of network parameters
- For general BNs
  - Exact inference is NP-hard
  - Approximate inference is NP-hard

- Most BNs are not polytrees. What to do?
  - Clustering (e.g., junction tree) algorithm (somewhat later) to make graph tree like and then use BP
  - Approximate inference (next slides)

Approximate Inference over BNs

Approximation

- We are going to approximate answers to queries such as the posterior (PX|E).
- Here we assume an intuitive of approximation:
  - The answer may be erroneous up to some amount
- Formally treated in PAC theory (Probably Approximately Correct) by parameters (δ,ε)
  - Confidence (quantified by δ) in that found solution maximally deviates from true solution up to ε
- PAC discussed in one of the next lectures

Approximate Inference in Bayesian Networks

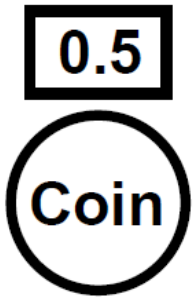
Monte Carlo algorithm

- Widely used to estimate quantities that are difficult to calculate exactly (Remember: for BNs NP-hardness)
- Randomized sampling algorithm
- Accuracy depends on the number of samples
- Two families
  - Direct sampling
  - Markov chain sampling

Inference by stochastic simulation

Basic idea:

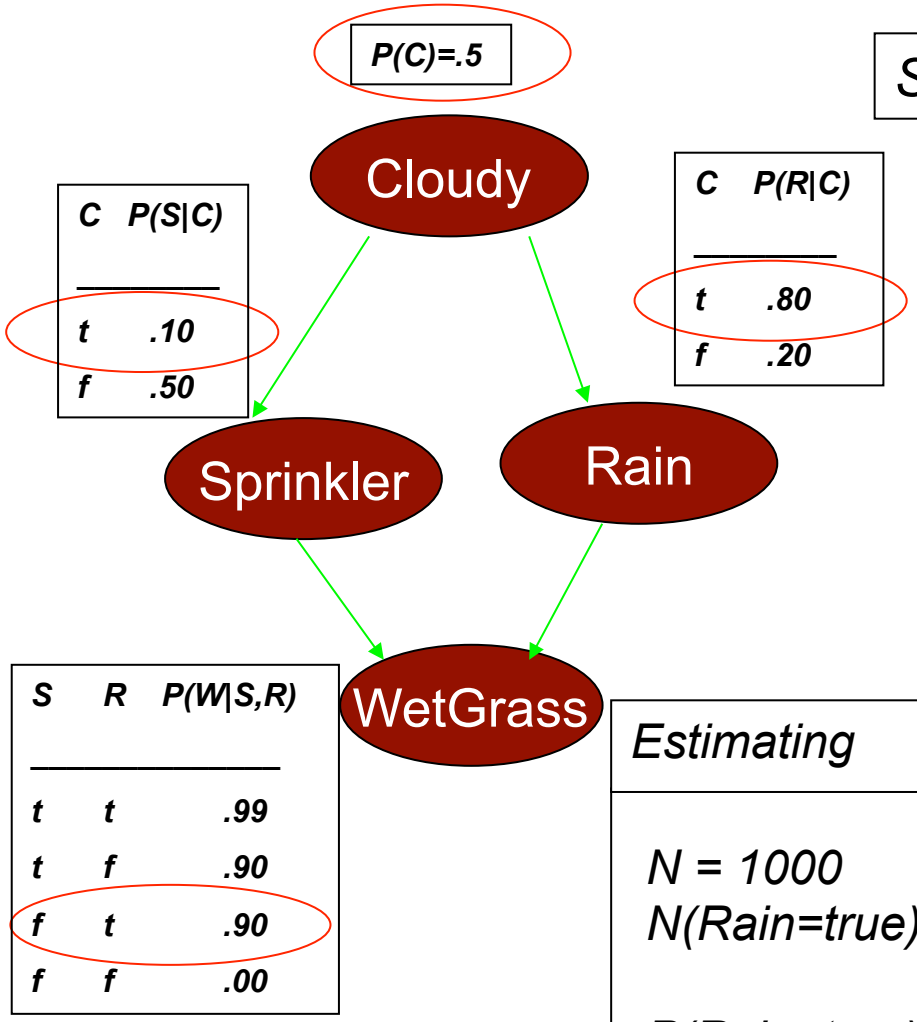
1) Draw N samples from a sampling distribution S
2) Compute an approximate posterior probability $\hat{P}$
3) Show this converges to the true probability P



Outline:

- Sampling from an empty network
- Rejection sampling: reject samples disagreeing with evidence
- Likelihood weighting: use evidence to weight samples
- Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

Example in simple case



Sampling

[Cloudy, Sprinkler, Rain, WetGrass]

[true, , ,]

[true, false, ,]

[true, false, true,]

[true, false, true, true]

Estimating

N = 1000 *(# generated samples)*

N(Rain=true) = N([_ , _ , true, _]) = 511
(# of samples for this event)

P(Rain=true) = 0.511 *(approximated probability)*

Sampling from empty network

- Generating samples from a network that has no evidence associated with it (*empty* network)
- Basic idea
  - sample a value for each variable in topological order
  - using the specified conditional probabilities

function PRIOR-SAMPLE(bn) **returns** an event sampled from bn
  inputs: bn, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  $\mathbf{x} \leftarrow$ an event with n elements
  for $i = 1$ **to** n **do**
    $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
      given the values of $Parents(X_i)$ in $\mathbf{x}$
  return $\mathbf{x}$

Properties

Probability that PRIORSAMPLE generates a particular event

$$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i)) = P(x_1 \ldots x_n)$$

i.e., the true prior probability

E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$

Then we have

$$\begin{aligned} \lim_{N \to \infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N \to \infty} N_{PS}(x_1, \ldots, x_n)/N \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

That is, estimates derived from PRIORSAMPLE are consistent

Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1 \ldots x_n)$

What if evidence is given?

- Sampling as defined above would generate cases that cannot be used

Rejection Sampling

- Used to compute conditional probabilities
- Procedure
  - Generating sample from prior distribution specified by the Bayesian Network
  - Rejecting all that do not match the evidence
  - Estimating probability

Rejection Sampling

$\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

function REJECTION-SAMPLING($X, \mathbf{e}, bn, N$) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{N}$, a vector of counts over X, initially zero

  for $j = 1$ to N **do**
    $\mathbf{x} \leftarrow$ PRIOR-SAMPLE(bn)
    if $\mathbf{x}$ is consistent with $\mathbf{e}$ **then**
      $\mathbf{N}[x] \leftarrow \mathbf{N}[x]+1$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{N}[X]$)

Rejection Sampling Example

- Let us assume we want to estimate P(Rain|Sprinkler = true) with 100 samples
- 100 samples
  - 73 samples => Sprinkler = false
  - 27 samples => Sprinkler = true
    - 8 samples => Rain = true
    - 19 samples => Rain = false
- P(Rain|Sprinkler = true) = NORMALIZE((8,19)) = (0.296,0.704)
- Problem
  - It rejects too many samples

Analysis of rejection sampling

$$
\begin{aligned}
\hat{\mathbf{P}}(X|\mathbf{e}) &= \alpha \mathbf{N}_{PS}(X,\mathbf{e}) && \text{(algorithm defn.)} \\
&= \mathbf{N}_{PS}(X,\mathbf{e})/N_{PS}(\mathbf{e}) && \text{(normalized by } N_{PS}(\mathbf{e})\text{)} \\
&\approx \mathbf{P}(X,\mathbf{e})/P(\mathbf{e}) && \text{(property of PriorSample)} \\
&= \mathbf{P}(X|\mathbf{e}) && \text{(defn. of conditional probability)}
\end{aligned}
$$

Hence rejection sampling returns consistent posterior estimates

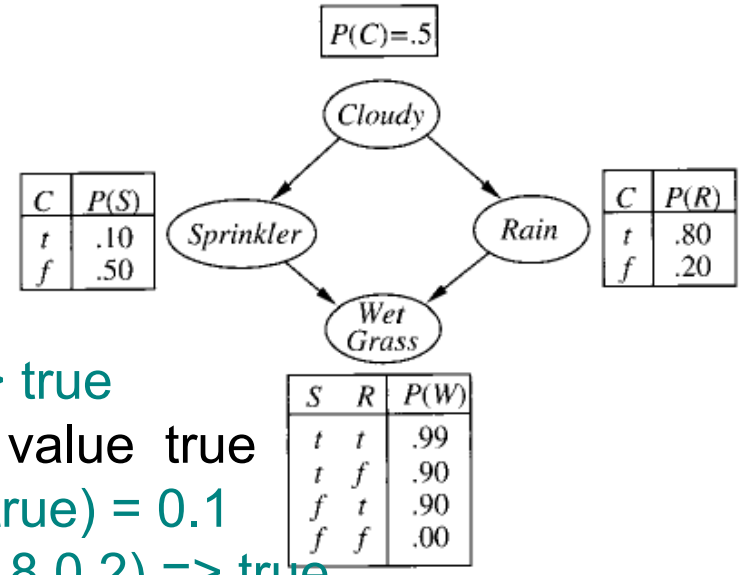
Problem: hopelessly expensive if $P(\mathbf{e})$ is small

$P(\mathbf{e})$ drops off exponentially with number of evidence variables!

Likelihood Weighting

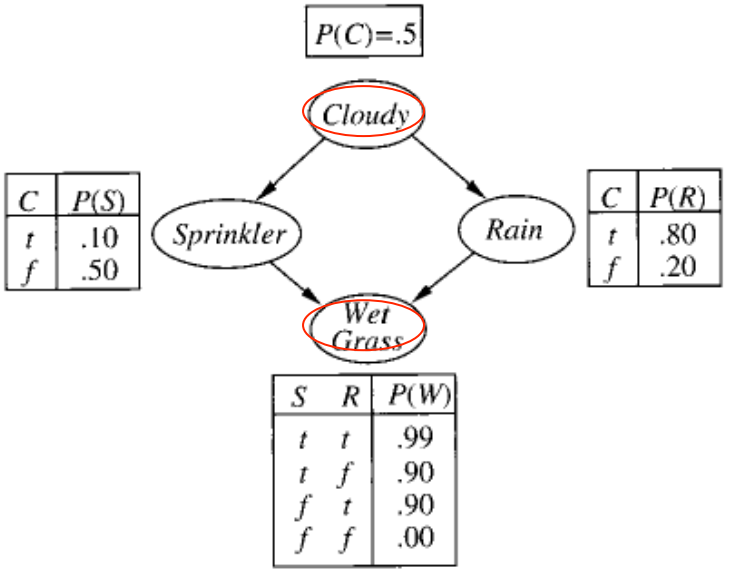
- Goal
  - Avoiding inefficiency of rejection sampling
- Idea
  - Generating only events consistent with evidence
  - Each event is weighted by likelihood that the event accords to the evidence

Likelyhood Weighting: Example



- **P**(Rain|Sprinkler=true, WetGrass = true)?
- Sampling
  - The weight is set to 1.0
  - Sample from **P**(Cloudy) = (0.5,0.5) => true
  - Sprinkler is an evidence variable with value true
    w ← w * P(Sprinkler=true | Cloudy = true) = 0.1
  - Sample from P(Rain|Cloudy=true)=(0.8,0.2) => true
  - WetGrass is an evidence variable with value true
    w ←w * P(WetGrass=true |Sprinkler=true, Rain = true) = 0.099
  - [true, true, true, true] with weight 0.099
- Estimating
  - Accumulating weights to either Rain=true or Rain=false
  - Normalize (= divide by sum of weights)

Likelyhood Weighting: Example



- **P**(Rain|Cloudy=true, WetGrass = true)?
- Sampling
  - Cloudy is an evidence
    w ← w * P(Cloudy = true) = 0.5
  - Sprinkler no evidence
    Sample from P(Sprinkler| Cloudy=true)=(0.1, 0.9) => false
  - Sample from P(Rain|Cloudy=true)=(0.8,0.2) => true
  - WetGrass is an evidence variable with value true
    w ←w * P(WetGrass=true |Sprinkler=false, Rain = true) = 0.45
  - [true, false, true, true] with weight 0.45

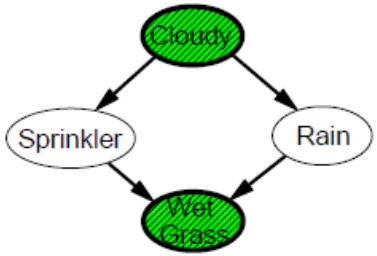
Likelihood analysis

Sampling probability for WeightedSample is

$$S_{WS}(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{l} P(z_i | parents(Z_i))$$

Note: pays attention to evidence in **ancestors** only

$\Rightarrow$ somewhere "in between" prior and posterior distribution



Weight for a given sample $\mathbf{z}, \mathbf{e}$ is

$$w(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{m} P(e_i | parents(E_i))$$

Weighted sampling probability is

$$\begin{aligned} & S_{WS}(\mathbf{z}, \mathbf{e}) w(\mathbf{z}, \mathbf{e}) \\ & = \prod_{i=1}^{l} P(z_i | parents(Z_i)) \prod_{i=1}^{m} P(e_i | parents(E_i)) \\ & = P(\mathbf{z}, \mathbf{e}) \text{ (by standard global semantics of network)} \end{aligned}$$

Hence likelihood weighting returns consistent estimates but performance still degrades with many evidence variables because a few samples have nearly all the total weight

Likelihood weighting

function LIKELIHOOD-WEIGHTING($X, \mathbf{e}, bn, N$) **returns** an estimate of $P(X|\mathbf{e})$

  local variables: $\mathbf{W}$, a vector of weighted counts over X, initially zero

  for $j = 1$ to N **do**

    $\mathbf{x}, w \leftarrow$ WEIGHTED-SAMPLE(bn)

    $\mathbf{W}[x] \leftarrow \mathbf{W}[x] + w$ where x is the value of X in $\mathbf{x}$

  return NORMALIZE($\mathbf{W}[X]$)

---

function WEIGHTED-SAMPLE($bn, \mathbf{e}$) **returns** an event and a weight

  $\mathbf{x} \leftarrow$ an event with n elements; $w \leftarrow 1$

  for $i = 1$ **to** n **do**

    if X_i has a value x_i in $\mathbf{e}$

      then $w \leftarrow w \times P(X_i = x_i \mid parents(X_i))$

      else $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$

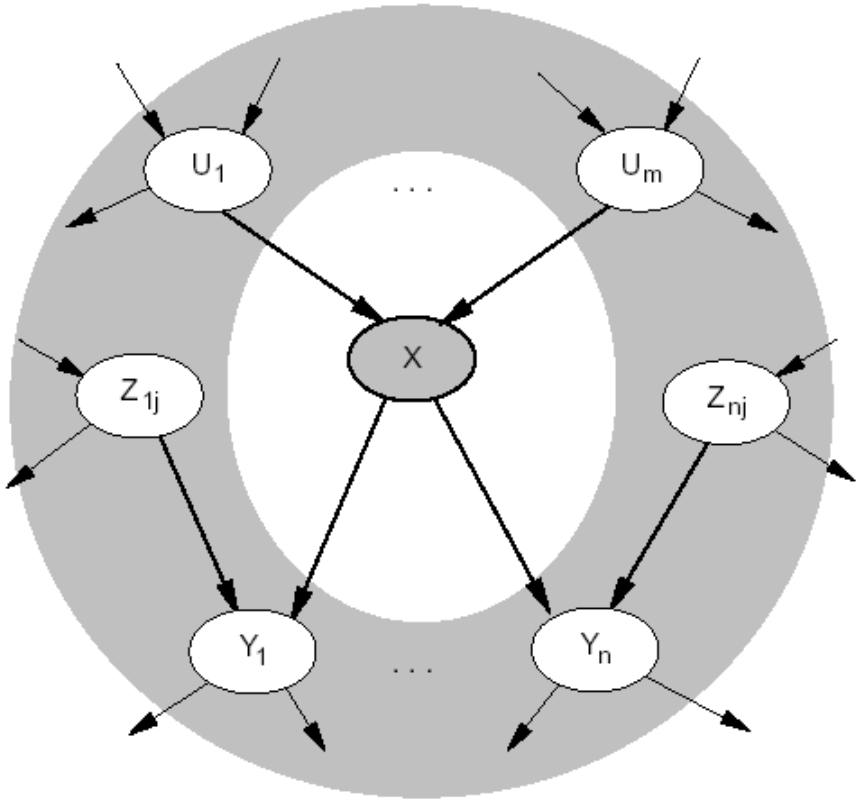
  return $\mathbf{x}$, w

Markov Chain Monte Carlo

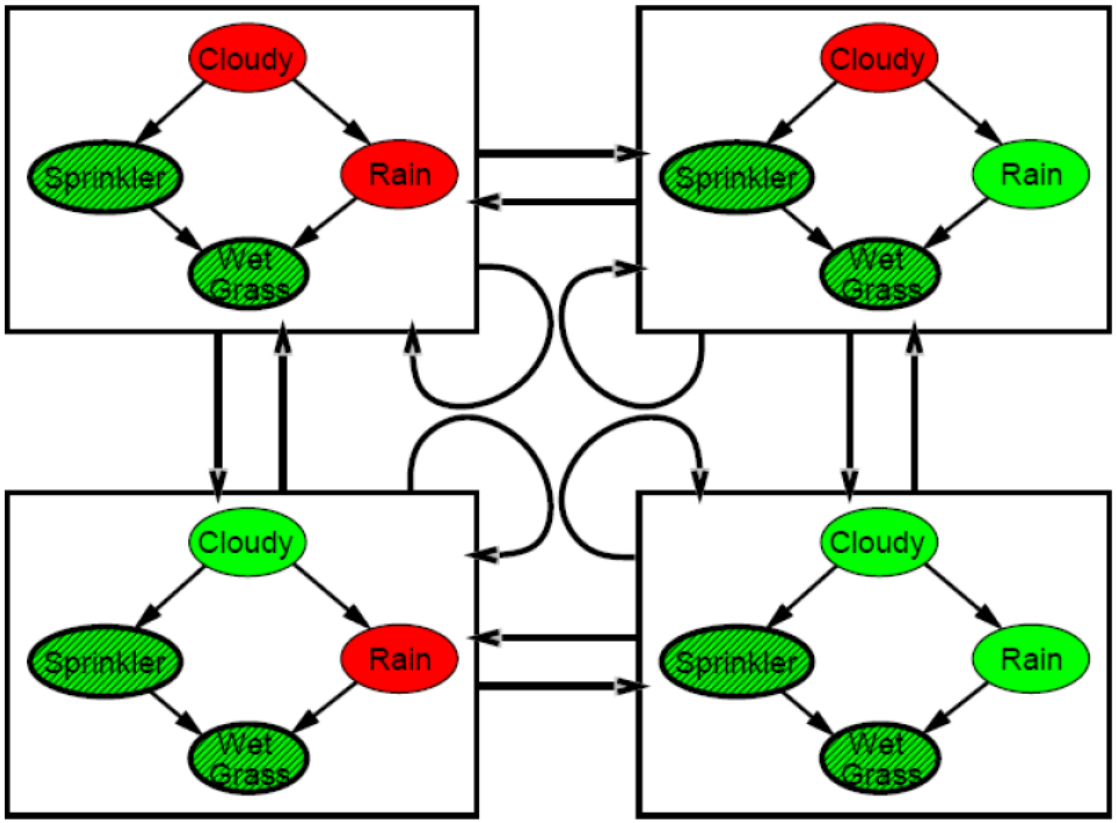
- Let's think of the network as being in a particular current state specifying a value for every variable
- MCMC generates each event by making a random change to the preceding event
- The next state is generated by randomly sampling a value for one of the non-evidence variables X_i, conditioned on the current values of the variables in the Markov blanket of X_i
- Likelihood Weighting only takes into account the evidences of the parents. (Problematic if evidence on leafs).

Markov Blanket

- Markov blanket: Parents + children + children's parents
- Node is conditionally independent of all other nodes in network, given its Markov Blanket



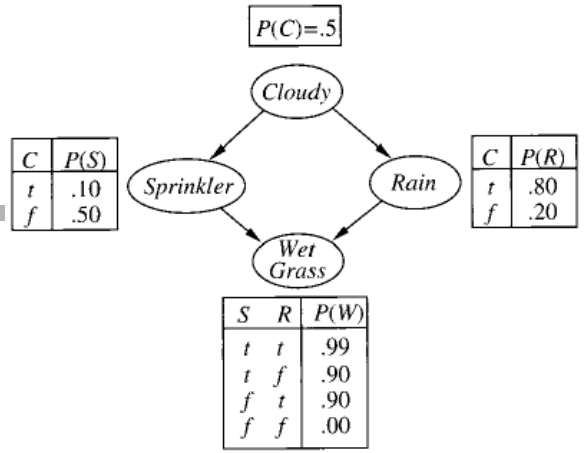
With $Sprinkler = true, WetGrass = true$, there are four states:



Wander about for a while, average what you see

Arrows describe transition probabilities; leads to a (the Markov) chain of states

Markov Chain Monte Carlo: Example



- P(Rain|Sprinkler = true, WetGrass = true) = ?
- Initial state is [true, true, false, true] (for [Cloudy,Sprinkler,Rain,WetGrass])

- The following steps are executed repeatedly:
  - Cloudy is sampled, given the current values of its Markov blanket variables
    So, we sample from P(Cloudy|Sprinkler= true, Rain=false)
    Suppose the result is Cloudy = false.
  - Now current state is [false, true, false, true] and counts are updated
  - Rain is sampled, given the current values of its Markov blanket variables
    So, we sample from P(Rain|Cloudy=false,Sprinkler=true, WetGrass=true)
    Suppose the result is Rain = true.
  - Then current state is [false, true, true, true]
- After all the iterations, let's say the process visited 20 states where Rain is true and 60 states where Rain is false then the answer of the query is NORMALIZE((20,60))=(0.25,0.75)

MCMC

"State" of network = current assignment to all variables.

Generate next state by sampling one variable given Markov blanket
Sample each variable in turn, keeping evidence fixed

function MCMC-ASK($X, \mathbf{e}, bn, N$) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{N}[X]$, a vector of counts over X, initially zero
    $\mathbf{Z}$, the nonevidence variables in bn
    $\mathbf{s}$, the current state of the network, initially copied from $\mathbf{e}$
  initialize $\mathbf{s}$ with random values for the variables in $\mathbf{Z}$
  for $j = 1$ to N **do**
    for each Z_i in $\mathbf{Z}$ **do**
      sample the value of Z_i in $\mathbf{s}$ from $\mathbf{P}(Z_i|mb(Z_i))$
        given the values of $MB(Z_i)$ in $\mathbf{s}$
      $\mathbf{N}[x] \leftarrow \mathbf{N}[x] + 1$ where x is the value of X in $\mathbf{s}$
  return NORMALIZE($\mathbf{N}[X]$)

Can also choose a variable to sample at random each time

Summary

- Bayesian networks provide a natural representation for (causally induced) conditional independence
- Topology + CPTs = compact representation of joint distribution
- Generally easy for domain experts to construct
- Exact inference by variable elimination
  - polytime on polytrees, NP-hard on general graphs
  - space can be exponential as well
- Exact Inference by belief propagation on polytress
- Approximate inference based on sampling and counting help to overcome complexity of exact inference