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Belief Revision with Bridging Axioms

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Abstract

Belief revision deals with the problem of changing a declaratively specified repository under potentially conflicting information. Usually, the problem is approached by providing postulates that specify intended constraints for the revision and constructing concrete revision operators fulfilling them. In the last 30 years since the start of formal belief revision with the work of AGM (Alchourron, Gärdenfors, and Makinson) roughly four construction principles were investigated and mutually interrelated: partial-meet, epistemic entrenchment, safe/kernel, and the possible worlds (model based) construction. The aim of this paper is to raise into the focus another construction principle relying on the idea of reinterpretation: Conflicts are explained by different use of symbols and conflict resolution is handled by choosing appropriate bridging axioms that relate the different readings. The main purpose of the paper is to argue that the reinterpretation-based approach is sufficiently general by showing how to equivalently formulate classical revision operators such as the operators of Weber, a natural variant of Weber, the operator of Satoh (= skeptical operator of Delgrande and Schaub) and the operator of Borgida with reinterpretation operators.

1 Introduction

Belief revision deals with the problem of changing a declaratively specified repository such as a belief base (an arbitrary set of sentences) or a belief set (a logically closed set of sentences) under a new piece of potentially conflicting information, called *trigger* in the following. This paper is going to discuss a class of operators in a special category of belief revision that—following the terminology of (Dalal 1988)—is termed "knowledge-base revision". The idea is to represent the knowledge of a domain in a *finite* belief base and define revision operators that do not depend on the syntax of the belief base but only on its semantics, i.e., its models.

In the last 30 years since the start of formal belief revision with the work of AGM (Alchourron, Gärdenfors, and Makinson, (Alchourrón, Gärdenfors, and Makinson 1985)), roughly four construction principles were investigated and mutually interrelated: The first one is that of partial-meet revision going back to AGM (Alchourrón, Gärdenfors, and Makinson 1985). The result is calculated by considering maximal subsets of the belief set consistent with the trigger and then intersecting a selection of them (hence the name partial-meet); another construction principle (applied mainly to belief bases) uses kernels, i.e., minimal sets of inconsistencies, in order to guide the contraction/revision. The third principle rests on ranking sentences w.r.t. an epistemic entrenchment relation. And the last construction principle approaches revision purely semantically, considering only possible worlds or more specifically, models, for the KB and the trigger, taking into consideration relations (orderings) on the possible worlds to guide the revision. Knowledge-base revision falls into this last category.

The aim of this paper is to raise into the focus another—though not completely orthogonal—construction principle relying on the idea of reinterpretation: Conflicts are explained by different use of symbols, and conflict resolution is handled by choosing appropriate bridging axioms that relate the different readings. As an example, consider an integration scenario where an ontology from an online library system (trigger) has to be integrated into an ontology of another online library system (the knowledge base). Here one faces the problem of ambiguous use of symbols. For example, the term "article" may be used in one library system for all entities that are published in the proceedings of a conference or in a journal, and in the other "article" may stand for entities published only in journals. So one reading of "article" has to be reinterpreted such that conflicts are resolved and such that the different readings are interrelated in a reasonable way. In this example, one would state declaratively with a bridging axiom that one reading of "article" is contained in the other reading (but not vice versa).

The properties of reinterpretation heavily depend on the types of bridging axioms that one is going to allow to guide the reinterpretation. Choosing different classes of bridging axioms leads to different classes of reinterpretation operators. In the example above, allowing only equivalence (in propositional logic: bi-implication) may lead to a loss of relations between the different readings of "article" whereas a use of subsumption (in propositional logic: implication) may preserve relevant relations between the readings.

Though the reinterpretation operators were built w.r.t. such integration scenarios as illustrated above, this paper argues that the reinterpretation based framework is sufficiently general to capture also classical revision operators. In particular, the main contribution of this paper is to show that one can find appropriate classes of bridging axioms such that some of the operators discussed in (Eiter and Gottlob 1992) can be equally represented as reinterpretation operators: The revision operator of Weber (Weber 1986) can be represented by full-meet revision on the set of bridging axioms that have the form of bi-implications. A natural variant of Weber revision can be represented as full-meet revision of bridging axioms having the form of implications. The operator of Satoh (Satoh 1988) (= skeptical operator of (Delgrande and Schaub 2003)) can be represented as partial-meet revision of the disjunctive closure of bi-implications, and the operator of Borgida (Borgida 1985) can be represented as partial-meet revision of bridging axioms that are in the disjunctive closure of bi-implications or that are literals.

The rest of the paper is structured as follows: After a short overview of related work in Sect. 2 and a recap of necessary terminology and concepts in Sect. 3, classical model-based belief-revision operators are introduced in Sect. 4. The class of reinterpretation operators is introduced in Sect. 5. The main equivalence results are stated and proved in Sect. 6. Section 7 concludes the paper with a short resume and an outlook. At the URL `http://tinyurl.com/jprcbuv` an extended version of this paper with proofs is provided.

2 Related Work

The reinterpretation framework uses bridging axioms to guide the resolution of conflicts. The most related work is that of Delgrande and Schaub (Delgrande and Schaub 2003) discussed below. But, additionally, there is a great deal of work in the general area of ontology change with which the reinterpretation framework presented in this paper shares the main motivations. A classification of different forms of ontology change operators with pointers to the literature is given by Flouris and colleagues (Flouris et al. 2008). Work in ontology mapping, ontology alignment, and mapping revision (Meilicke and Stuckenschmidt 2009; Qi, Ji, and Haase 2009) is related to the approach described in this paper, as it investigates adequate constructions of mappings which are generalizations of bridging axioms. A more recent approach for mapping management in the paradigm of ontology-based data access is given in (Lembo et al. 2015). A more recent approach to ontology revision is developed in (Benferhat et al. 2014).

The reinterpretation approach is symbol-oriented. A different symbol-oriented approach is described by Lang and Marquis (Lang and Marquis 2010). Their revision operators are not based on bridging axioms but use the concept of forgetting (Lin and Reiter 1994). As exemplified by the Weber operator (Weber 1986) (see Sect. 4), there are strong connections between these approaches.

3 Preliminaries

This section shortly describes concepts and notations used in this paper. Though the idea of reinterpretation unfolds its full effect in expressive logics such as first-order logic (FOL) or description logics, this paper focusses on propositional logic.

The power set of a set X is denoted $\mathrm{Pow}(X) = \{Y \mid Y \subseteq X\}$. In the whole paper $\mathcal{P}$ denotes a set of propositional symbols. These are denoted by p, q, r etc. and variants with subscripts and primes. $\mathrm{Fml}(\mathcal{P})$ is the set of propositional formulae α over $\mathcal{P}$ given by the following grammar:

$$\alpha ::= p \mid \neg\alpha \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha) \mid (\alpha \rightarrow \alpha) \mid (\alpha \leftrightarrow \alpha) \mid \bot \mid \top$$

$\alpha, \beta \dots$ are used as meta-variables for propositional formulae. Any finite set of propositional formulae is called a *knowledge base* and is denoted by B or indexed or primed variants. $\bigwedge B$ abbreviates $(\alpha_1 \wedge \alpha_2) \wedge \dots) \wedge \alpha_n$ for some fixed ordering of all elements α_i of B. The set of propositional symbols in B is abbreviated as $\mathrm{symb}(B)$. The semantics is defined as usual by truth-tables based on *interpretations*. $\mathrm{Int}(\mathcal{P}) = \{1, 0\}^{\mathcal{P}}$ denotes the set of *interpretations*, i.e., functions from $\mathcal{P}$ to the set of truth values $1, 0$. Interpretations are denoted by $\mathcal{I}, \mathcal{J}$ and indexed and primed variants. The modelling relation and the entailment relation are defined as usual and are both denoted by $\models$. $[\![B]\!]$ denotes the set of models of B. Actually, for the definition of some operators, interpretations will be identified with the set of symbols which are assigned the value 1. For example, let $\mathcal{I}$ be an interpretation over $\mathcal{P} = \{p, q, r\}$ with $\mathcal{I}(p) = 1$, $\mathcal{I}(q) = 0, \mathcal{I}(r) = 1$, then $\mathcal{I}$ is identified with the set $\{p, r\}$.

The definition of reinterpretation operators requires two disjoint sets of propositional symbols, a set $\mathcal{P}$, also called the *public symbol set* and a set $\mathcal{P}' := \{p' \mid p \in \mathcal{P}\}$ called the *internal symbol* set. The paper relies on the following sets of *bridging axioms*: *Bimpl* $= \{p \leftrightarrow p' \mid p \in \mathcal{P}\}$ and *Impl* $= \{p \rightarrow p', p' \rightarrow p \mid p \in \mathcal{P}\}$.

The set of *consequences* of B over the set of propositional symbols S is defined as $\mathrm{Cn}^S(B) = \{\alpha \in \mathrm{Fml}(S) \mid B \models \alpha\}$. If Cn is used without a superscript, then the consequences have to be understood with respect to the maximal set of propositional symbols discussed in the context, in our case this is mostly the symbol set $\mathcal{P} \cup \mathcal{P}'$. If two sets B_1 and B_2 have the same sets of consequences of formulae in $\mathrm{Fml}(S)$, we write $B_1 \equiv_S B_2$. Some of the equivalence results heavily depend on the disjunctive closure (Hansson 1999). This is a closure operator that is rougher than the usual consequence but finer than the identity closure. The *disjunctive closure* $\overline{B}$ of a knowledge base B is defined as follows: $\overline{B} = \{\beta_1 \vee \dots \vee \beta_n \mid \beta_i \in B, n \in \mathbb{N} \setminus \{0\}\}$.

Reinterpretation operators are defined with *dual remainder sets*, which are similar to the concept of remainder sets used in the classical paper of Alchourrón, Gärdenfors and Makinson (AGM) (Alchourrón, Gärdenfors, and Makinson 1985) but which are applicable also for logics not allowing for arbitrary negation. As we are going to consider also the more general case of *multiple revision*, i.e. revision with sets Y of sentences as triggers, we define the following notions for this general case. The special case of singletons $Y = \{\alpha\}$ then covers the case of a trigger that is a sentence α. *The dual remainder sets modulo* Y are defined as follows:

$$X \in B \top Y \quad \text{iff} \quad \begin{array}{l} X \subseteq B \text{ and } X \cup Y \text{ is consistent and} \\ \text{for all } X' \subseteq B \text{ with } X \subsetneq X' \\ \text{the set } X' \cup Y \text{ is not consistent} \end{array}$$

Let B be a knowledge base. An *AGM-selection function* γ for B is a function $\gamma : \text{Pow}(B) \longrightarrow \text{Pow}(B)$, such that for all sets of formulae Y the following holds:
1. If $B \top Y \neq \emptyset$, then $\emptyset \neq \gamma(B \top Y) \subseteq B \top Y$;
2. else $\gamma(\emptyset) = \{B\}$.

With these notions one can define multiple partial-meet revision for belief bases (see, e.g., (Hansson 1999)). Given any set B, a AGM-selection function γ for B, and any set Y of formulae, *multiple partial-meet base revision* is defined as: $B *_\gamma Y = \bigcap \gamma(B \top Y) \cup Y$. We let $B *_\gamma \alpha = B *_\gamma \{\alpha\}$.

The function pr_i is the projection of n-ary ($i \leq n$) vectors to their i^{th} argument.

As we are going to define quite a lot of change operators, we make the following notational convention: all revision operators are denoted by $*$, possibly with super- and subscripts. All reinterpretation operators are denoted by the symbol $\circ$, possibly with super- and subscripts.

4 Model-Based Belief Revision

The main idea of *model-based* belief revision is to let the revision be driven only by the models of the knowledge base and of the trigger. With this approach the concrete sentential representation of the belief base becomes irrelevant (hence it is called knowledge base), and, in fact, Dalal (Dalal 1988) considered this syntax-insensitivity as the essence of *knowledge-base* revision in contrast to belief-set revision according to AGM (Alchourrón, Gärdenfors, and Makinson 1985) or belief-base revision (Hansson 1991).

Though quite many different model-based operators exist, the core idea for the revision is the same: The models of the revision are those models of the trigger that are minimal w.r.t. some appropriate (pre-, partial, or total) order or, more specifically, a distance function. As shown by Katsuno and Mendelzon (Katsuno and Mendelzon 1992), there are strong connections between revision operators based on orders (of a specific kind) and the postulates they fulfill.

All model-based operators that are in the focus of this paper are defined on the basis of minimal difference between models of the knowledge base and the trigger. Minimal difference in turn is explicated by using—in some or other form—the symmetric difference of models represented as sets. The *symmetric difference* for any pair of sets A, B is defined as $A \Delta B = A \setminus B \cup B \setminus A$. Let X_1, X_2 be sets of sentences. $\Delta^{min}(X_1, X_2)$ is the set of inclusion-minimal symmetric differences between models of X_1 and X_2.

$$\Delta^{min}(X_1, X_2) = min_\subseteq \{\mathcal{I} \Delta \mathcal{J} \mid \mathcal{I} \in [\![X_1]\!], \mathcal{J} \in [\![X_2]\!]\}$$

If one considers a set of sentences X_1 with exactly one model $\mathcal{I}$ in the first argument, then one gets as special case $\Delta^{min}(\mathcal{I}, X_2)$ which is the set of inclusion-minimal symmetric differences between $\mathcal{I}$ and models of X_2. (Remember that $\mathcal{I}$ is identified with the set of proposition symbols that are true according to $\mathcal{I}$).

$$\Delta^{min}(\mathcal{I}, X_2) \quad = \quad min_\subseteq \{\mathcal{I} \Delta \mathcal{J} \mid \mathcal{J} \in [\![X_2]\!]\}$$

The set $\Omega(X_1, X_2)$ describes those propositional variables that are involved in a minimal difference between a model of X_1 and a model of X_2.

$$\Omega(X_1, X_2) \quad = \quad \bigcup \Delta^{min}(X_1, X_2)$$

In the symmetric difference operator Δ, information regarding the origins of the elements is lost. This loss is mitigated within the following non-commutative definition of the symmetric difference: $\Delta_\pm(A, B) = (A \setminus B, B \setminus A)$. For any set of sentences X_1, X_2 define

$$\Delta_\pm(X_1, X_2) = \{(\mathcal{I} \setminus \mathcal{J}, \mathcal{J} \setminus \mathcal{I}) \mid \mathcal{I} \in [\![X_1]\!], \mathcal{J} \in [\![X_2]\!]\}$$

Now consider a subset of these sets which are minimal w.r.t. the cartesian product order $<_{\subseteq \times \subseteq}$ defined as usual by $(A, B) <_{\subseteq \times \subseteq} (C, D)$ iff $A \subseteq C$ and $B \subseteq D$.

$$\Delta^{min}_\pm(X_1, X_2) \quad = \quad min_{<_{\subseteq \times \subseteq}}(\Delta_\pm(X_1, X_2))$$

The adapted operator for $\Omega(\cdot, \cdot)$ is denoted $\Omega_\pm(X_1, X_2)$ which is defined as the pair of two sets: the first (second) argument consists of all propositional variables contained in the first (second) argument of some pair in $\Delta^{min}_\pm(X_1, X_2)$.

$$\Omega_\pm(X_1, X_2) = \Big(\bigcup \{Y_1 \mid \exists Y_2 : (Y_1, Y_2) \in \Delta^{min}_\pm(X_1, X_2)\}, \bigcup \{Y_2 \mid \exists Y_1 : (Y_1, Y_2) \in \Delta^{min}_\pm(X_1, X_2)\} \Big)$$

The Satoh revision operator $*_S$ (Satoh 1988) defines the models of the revision result as those models of the trigger for which there is a model of the knowledge base with minimal symmetric difference.

$$[\![B *_S \alpha]\!] \quad = \quad \{\mathcal{I} \in [\![\alpha]\!] \mid \text{There is } \mathcal{J} \in [\![B]\!] \text{ with } \mathcal{I} \Delta \mathcal{J} \in \Delta^{min}(B, \alpha)\}$$

A natural weakening of this operator is the following one, which is coined *weak Satoh revision* here.

$$[\![B *_{wkS} \alpha]\!] \quad = \quad \{\mathcal{I} \in [\![\alpha]\!] \mid \text{There is } \mathcal{J} \in [\![B]\!] \text{ with } \mathcal{I} \Delta_\pm \mathcal{J} \in \Delta^{min}_\pm(B, \alpha)\}$$

Note that in both Satoh revision operators, minimality concerns the whole set of models of the knowledge base. In contrast to this, the operator of Borgida (Borgida 1985) considers for each model of the knowledge base the models of the trigger that are minimally distant. Borgida revision is defined as follows: If $B \cup \{\alpha\}$ is consistent, then $[\![B *_B \alpha]\!] = [\![B \cup \{\alpha\}]\!]$. Otherwise,

$$[\![B *_B \alpha]\!] = \bigcup_{\mathcal{I} \in [\![B]\!]} \{\mathcal{J} \in [\![\alpha]\!] \mid \mathcal{I} \Delta \mathcal{J} \in \Delta^{min}(\mathcal{I}, \{\alpha\})\}$$

The operator of Weber (Weber 1986) puts all those models of the trigger into the revision result for which there is a knowledge base model differing at most in the propositional variables involved in a minimal difference. In the trivial case where $B \cup \{\alpha\}$ is consistent the definition is $[\![B *_B \alpha]\!] = [\![B \cup \{\alpha\}]\!]$. Else:

$$[\![B *_W \alpha]\!] \quad = \quad \{\mathcal{J} \in [\![\alpha]\!] \mid \text{There is } \mathcal{I} \in [\![B]\!] \text{ s.t. } \mathcal{I} \setminus \Omega(B, \alpha) = \mathcal{J} \setminus \Omega(B, \alpha)\}$$

A natural variant of the Weber operator uses the non-commutative definition of symmetric difference. In lack of a better name this operator is coined the *weak Weber* operator.

$$[\![B *_{wkW} \alpha]\!] = \{\mathcal{J} \in [\![\alpha]\!] \mid \text{There is } \mathcal{I} \in [\![B]\!] \text{ s.t. } \mathcal{I} \setminus pr_1(\Omega_\pm(B, \alpha)) = \mathcal{J} \setminus pr_2(\Omega_\pm(B, \alpha))\}$$

5 Reinterpretation-Based Revision

The general idea of reinterpretation-based revision for a knowledge base B and a trigger α is to trace back the conflict between B and the trigger α to an ambiguous use of some of the common symbols. So, the idea for resolving the conflict is to assume in the first place how the different uses of the symbols are related, stipulating the relations explicitly as a set of bridging axioms BA, and then applying a classical revision strategy on BA as the knowledge base to be revised.

Example 1. *Assume that the sender of the trigger has a strong notion of article defined to be those entities published in a journal. The sender of the trigger has stored the information that some entity b is not an article in her KB because she has acquired the knowledge that b is not published in a journal. Assume that the information $\neg Article(b)$ is represented in propositional logic by the literal $\alpha = \neg q$. This bit of information α is send to the holder of the KB B. The holder of B has a weaker notion of article—defining them as entities published either in the proceedings of a conference or in a journal. In her KB the same entity b is stated to be an article. So B entails q. Now, a conflict resolution strategy is to completely separate the readings of all symbols by renaming all the ones in B with primed versions, resulting in an "internalized" KB B'. In particular, q becomes q' in the receiver's KB. Then, guesses on the interrelations of the different readings are postulated by bridging axioms. The type of reinterpretation depends on the class of initial bridging axioms. So for example, considering bi-implications would lead to stipulations of axioms $p \leftrightarrow p'$, actually stating that the reading of p and p' is the same. The resolution of the conflicts between B and α requires not to include $q \leftrightarrow q'$, as this entails an inconsistency.*

Considering more fine-grained sets of bridging axioms such as implications $p \rightarrow p'$ and $p' \rightarrow p$ leads to more fine-grained solutions. In this example, the reinterpretation result could contain $q \rightarrow q'$ (articles as used in the trigger are articles as used in B) but not $q' \rightarrow q$ (articles in the sense of B are articles in the sense of α).

The parameterized equation $B \circ \alpha = BA * (B' \cup \{\alpha\})$ illustrates the general strategy for reinterpretation. The first parameter is the set of initial bridging axioms: BA contains sentences over $\mathcal{P} \cup \mathcal{P}'$ relating the meaning of the symbols $\mathcal{P}$ (associated with the sender) with those in $\mathcal{P}'$ (associated with the receiver). A very simple example of a bridging axiom is the bi-implication $p \leftrightarrow p'$ stating that the reading of p in the knowledge base is actually the same as the reading in the trigger. The trigger is the union of the original trigger and the internalized version of the original knowledge base B'. The second parameter in the schema is a classical base revision function $*$. In this paper, only partial-meet revision on arbitrary finite KBs is considered as instance of $*$.

Technically, reinterpretation-based revision is similar to base-generated revision (Hansson 1999) which combines the benefits of belief base revision and belief set revision, namely: the benefit of having a finite (and hence implementable) resource and the benefit of syntax-insensitivity. The difference relies in the special type of the generating base one uses, namely a set of bridging axioms. As the reinterpretation-based approach is not sentence-oriented but symbol-oriented it can be applied to various logics such as description logics (Eschenbach and Özçep 2010).

Four different classes of reinterpretation operators fitting the above scheme are those based on bi-implications as bridging axioms and those based on implications, both in turn considered per se or w.r.t. the disjunctive closure.

Definition 1. *Let B be a knowledge base, α be a formula, and γ be a selection function for Impl ($\overline{Impl}$, Bimpl, $\overline{Bimpl}$, resp.) and $*_\gamma$ be a partial-meet revision operator for Impl ($\overline{Impl}$, Bimpl, $\overline{Bimpl}$, resp.). 1. The* implication-based, *2. the* disjunctively closed implication-based, *3. the* bi-implication based, *and 4. the* disjunctively closed bi-implication based reinterpretation operators *are defined as follows*

$$
\begin{aligned}
B \circ_\gamma^{\rightarrow} \alpha &= \mathit{Impl} *_\gamma (B' \cup \{\alpha\}) \\
B \circ_\gamma^{\overline{\rightarrow}} \alpha &= \overline{\mathit{Impl}} *_\gamma (B' \cup \{\alpha\}) \\
B \circ_\gamma^{\leftrightarrow} \alpha &= \mathit{Bimpl} *_\gamma (B' \cup \{\alpha\}) \\
B \circ_\gamma^{\overline{\leftrightarrow}} \alpha &= \overline{\mathit{Bimpl}} *_\gamma (B' \cup \{\alpha\})
\end{aligned}
$$

If γ is the identity function, then γ can be dropped and the resulting operators are called skeptical reinterpretation operators *(using the terminology of (Delgrande and Schaub 2003)). If γ is such that $|\gamma(X)| = 1$ for all X, then we talk of* choice reinterpretation *operators.*

Another reinterpretation operator—which does not fit into the homogeneous scheme of Definition 1 and hence is defined separately—is termed *literal-supported* reinterpretation operator. It uses the notion of a bridging axiom in a very tolerant way. Concretely, the operator uses the following set of bridging axioms:

$$\mathit{Bimpl}^+ = \overline{\mathit{Bimpl}} \cup \overline{\{p', \neg p' \mid p \in \mathcal{P}\}}$$

So, next to the disjunctive closure of bi-implications it contains the disjunctive closure of literals in the internal vocabulary.

Definition 2. *Let γ be an AGM-selection function for Bimpl$^+$ and $*_\gamma$ a multiple partial-meet revision operator for Bimpl$^+$. Then define the* literal-supported *reinterpretation operators by: $B \circ_\gamma^{lit} \alpha = \mathit{Bimpl}^+ *_\gamma (B' \cup \{\alpha\})$.*

Note that all results of all the reinterpretation operators contain bridging axioms and hence are not contained in $\mathrm{Fml}(\mathcal{P})$ but $\mathrm{Fml}(\mathcal{P} \cup \mathcal{P}')$. Accordingly, reinterpretation operators are not genuine revision operators. As the plan of the paper is to show that with reinterpretation operators different classical belief-revision operators can be simulated, the consequences of the reinterpretation result are restricted to formulae in the public vocabulary, i.e., to formulae in $\mathrm{Fml}(\mathcal{P})$. So, what is going to be shown in the following is that for the revision operators $*$ mentioned before one can find a reinterpretation operator $\circ_*$ such that for all B and α one has $B * \alpha \equiv_{\mathcal{P}} B \circ_* \alpha$.

For a discussion of the use of reinterpretation operators and the postulates they fulfill we refer the reader to (Eschenbach and Özçep 2010; Özçep 2008; Özçep 2012). For a discussion of a genuine "representation result" which describes a set of postulates characterizing a class of (reinterpretation) operators we refer the reader to (Özçep 2012).

It should be stressed already here that all reinterpretation operators considered in this paper are motivated by a specific integration scenario: The information in the KB and that in the trigger are over the same domain; there is trust in the information stemming from the sender of the trigger and there is a clear evidence that the symbols in the KB and in the trigger are strongly related though they may differ. This is the reason why, in this paper, only special kinds of bridging axioms are considered where one reading p is related to another related reading p'. Clearly, one can consider further bridging axioms that go beyond the resolution of ambiguities; for example, one may consider also bridging axioms that relate synonymous symbols (say "beverage" and "drink"). But nothing prevents the general reinterpretation framework from using these kinds of bridging axioms. Of course, what is required then is a knowledge engineering step (based on heuristics, say) regarding the potential conflicts in a given integration scenario in order to find an appropriate initial set of bridging axioms.

The idea of reinterpretation is used implicitly in the operators of Delgrande and Schaub (DS) (Delgrande and Schaub 2003), but not from the perspective of disambiguating symbols, rather using the bridging axioms as helper axioms for the revision. Moreover, though the conflict resolution is similar to reinterpretation it is not the same.

DS revision operators are defined using the notion of a belief extension. We describe here only their general framework for revision (and not that of parallel revision and contraction in a belief change scenario.)

Definition 3. *(Delgrande and Schaub 2003) Given B and α, a* belief extension *is defined as a set of the form* $\mathrm{Cn}^{\mathcal{P}}(B' \cup \{\alpha\} \cup \mathit{Bim}_i)$ *where* $\mathit{Bim}_i \in \mathit{Bimpl}\top(B' \cup \{\alpha\})$. *If no such* Bim_i *exists, then* $\mathrm{Fml}(\mathcal{P})$ *is the only belief extension. The family of all belief extensions is denoted by* $(E_i)_{i\in I}$.

A DS-selection function c is defined for I as $c(i) \in I$. So it corresponds to AGM-selection functions that select exactly one element.

Based on these notions, choice revision $*^c_{\mathrm{DS}}$, which selects exactly one belief extension, and skeptical revision, which selects all belief extensions, can be defined.

Definition 4. *(Delgrande and Schaub 2003) Given a KB B, a formula α and $(E_i)_{i\in I}$ the set of all belief extensions and a selection function over I with $c(I) = k$, the* choice revision $*^c_{DS}$ *and skeptical revision* $*_{DS}$ *are defined as follows:*
$B *^c_{DS} \alpha = E_k$ *(for* $c(I) = k$*) and* $B *_{DS} \alpha = \bigcap_{i\in I} E_i$

The revision results for $*^c_{\mathrm{DS}}, *_{\mathrm{DS}}$ are not finite. But Delgrande and Schaub (Delgrande and Schaub 2003) show that there are equivalent operators that have finite revision results using a polarity flipping operator.

A theorem that is relevant for the equivalence results of this paper is proven in (Delgrande and Schaub 2003) as Corollary 4.8, stating that skeptical DS revision is nothing else than Satoh revision.

Theorem 1 ((Delgrande and Schaub 2003))**.** *Skeptical DS revision is Satoh revision:* $[\![B *_{DS} \alpha]\!] = [\![B *_S \alpha]\!]$.

The idea of using belief extensions can also be used for other sets than bi-implications. This may result in the following definitions as given in (Özçep 2012). A set $\mathrm{Cn}^{\mathcal{P}}(B' \cup \{\alpha\} \cup X)$ is an *implication-based belief extension* iff $X \in \mathit{Impl}\top(B' \cup \{\alpha\})$. Let $(\mathit{Im}_i)_{i\in I}$ be the set of all implication-based consistent belief extensions for B and α and c be a selection function for I with $c(I) = k$. The new operators are defined as follows:

Definition 5. *The* implication-based choice revision $*^{c,\rightarrow}_{DS}$ *and the* implication-based skeptical revision $*^{\rightarrow}_{DS}$ *are defined by:* $B *^{c,\rightarrow}_{DS} \alpha = \mathit{Im}_k$ *(for* $c(I) = k$*) and* $B *^{\rightarrow}_{DS} \alpha = \bigcap_{i\in I} \mathit{Im}_i$

Also for these operators a finite representation theorem follows using an adapted flipping operator (Özçep 2012). Based on the finite representation result with partial flipping operators, it is possible to show a theorem corresponding to Thm. 1, namely that skeptical implication-based revision $*^{\rightarrow}_{\mathrm{DS}}$ actually is the same as weak Satoh revision.

Theorem 2. $[\![B *^{\rightarrow}_{DS} \alpha]\!] = [\![B *_{wkS} \alpha]\!]$

6 Equivalence Results

The first equivalence result of this paper states that Satoh revision can be represented by disjunctively closed bi-implication-based reinterpretation operators $\circ^{\overline{\leftrightarrow}}_{\gamma}$. For the proof one shows that skeptical DS revision can be represented by an operator $\circ^{\leftrightarrow}_{\gamma}$ with a simple selection function γ, and then uses Theorem 1.

Theorem 3. *Skeptical DS revision $*_{DS}$ can be represented by an operator $\circ^{\overline{\leftrightarrow}}_{\gamma}$ where γ is defined independently of B (and α). That is, there is a selection function γ s.t. for any KB B and formula α:* $[\![B *_{DS} \alpha]\!] = [\![\mathrm{Cn}^{\mathcal{P}}(B \circ^{\leftrightarrow}_{\gamma} \alpha)]\!]$.

The function $\gamma = \gamma_1$ used in this and the next theorem is $\gamma_1(H) = \{\ X \in H \mid X \cap \mathit{Bimpl}$ is maximal in $\{X' \cap \mathit{Bimpl} \mid X' \in H\}\ \}$. As a corollary we get:

Theorem 4. *Satoh revision can be simulated by disjunctively closed bi-implication-based reinterpretation operators $\circ^{\overline{\leftrightarrow}}_{\gamma}$: There is a selection function γ s.t. for any KB B and formula α:* $[\![B *_S \alpha]\!] = [\![\mathrm{Cn}^{\mathcal{P}}(B \circ^{\overline{\leftrightarrow}}_{\gamma} \alpha)]\!]$.

Similar observations as for the theorems above also lead to the representation of implication-based skeptical DS revision and weak Satoh revision by disjunctively closed implication-based reinterpretation.

Theorem 5. *Implication-based skeptical DS revision and weak Satoh revision can be simulated by disjunctively closed implication-based reinterpretation operators $\circ^{\overline{\rightarrow}}_{\gamma}$: There is a selection function γ such that for any KB B and formula α it holds that* $[\![B *^{\rightarrow}_{DS} \alpha]\!] = [\![B *_{wkS} \alpha]\!] = [\![\mathrm{Cn}^{\mathcal{P}}(B \circ^{\overline{\rightarrow}}_{\gamma} \alpha)]\!]$.

Here $\gamma = \gamma_2$ is the same as γ_1 except that one uses *Impl* instead of *Bimpl*.

Weber revision is quite similar to Satoh, but it is more tolerant w.r.t. the models to be taken into account in the revision result. Dually, this tolerance w.r.t. the models means more skepticism regarding the sentences to keep in the revision result. Actually, this is reflected in the following theorem which says that Weber revision can be simulated by bi-implication-based reinterpretation: That is, in contrast to

Satoh revision, the set of bi-implications is not exploited further w.r.t. logical consequences within the disjunctive closure. Moreover, as there is no additional closure of the bridging axioms, even full meet revision can be used, i.e., γ can be chosen as the identity function.

Theorem 6. *Weber revision can be represented by skeptical bi-implication-based reinterpretation $\circ^{Bimpl}$, i.e., for any KB B and formula α: $[\![B *_W \alpha]\!] = [\![\mathrm{Cn}^{\mathcal{P}}(B \circ^{\leftrightarrow} \alpha)]\!]$.*

With a similar argument one can show:

Theorem 7. *The weak Weber operator can be represented by skeptical implication-based reinterpretation.*

Borgida revision is special in the sense that it considers the minimal symmetrical difference of models not globally, but locally for each model of the knowledge base. This model dependency can be simulated by literal-supported reinterpretation which allows the use of arbitrary primed literals as bridging axioms and thus allows the construction of arbitrary models of the knowledge base.

Theorem 8. *Borgida revision can be represented by literal-supported reinterpretation: There is γ for Bimpl$^+$ such that for any KB B and formula α: $[\![B *_B \alpha]\!] = [\![\mathrm{Cn}^{\mathcal{P}}(B \circ^{lit}_{\gamma} \alpha)]\!]$.*

7 Conclusion

Though all reinterpretation operators have been developed mainly for ontology integrations scenarios, they provide a sufficiently general framework for investigating classical belief-revision operators—at least this has been shown for five classical belief-revision operators that are defined purely semantically. In particular, the reinterpretation framework allows to compare classical revision from a different angle (similar to base-generated revision) and to get a better understanding of their commons and their differences.

As future work, the study of reinterpretation operators is planned to be extended by considering in a more systematically way the kinds of bridging axioms one is going to allow. For example, the literals used as bridging axioms for the representation of Borgida revision do not really bridge the meanings of symbols. So, the question arises whether for more restricted classes of bridging axioms a representation of Borgida revision is possible. This whole study then has to be lifted to the more demanding scenario where the knowledge base and the trigger are expressed in more expressive logics such as description logics or first-order logic.

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Appendix: Proofs

Additional Concepts and Lemmata

The following space saving abbreviations for $p \in \mathcal{P}$ are used throughout this appendix: $\overleftrightarrow{p} = p \leftrightarrow p'$, $\overrightarrow{p} = p \rightarrow p'$ and $\overleftarrow{p} = p' \rightarrow p$. We remind the reader of the abbreviations $\mathit{Bimpl} = \{\overleftrightarrow{p} \mid p \in \mathcal{P}\}$ and $\mathit{Impl} = \{\overrightarrow{p}, \overleftarrow{p} \mid p \in \mathcal{P}\}$.

In order to prove Theorem 2 we need a similar result in finite representability as given by (Delgrande and Schaub 2003) for skeptical DS revision. So we first define the relevant notions of flipping and partial flipping here.

Let Bim_i be a set of bi-implications. For any knowledge base B the formula $\lceil B \rceil_i$ results from B by substituting every propositional variable $p \in \mathcal{P}$ not occurring in Bim_i by its negation $\neg p$ and then building a formula by applying $\bigwedge$. The main observation for the finite representation result of (Delgrande and Schaub 2003) is the following observation:

Lemma 1 ((Delgrande and Schaub 2003))**.** *Let* $\mathit{Bim}_i \in \mathit{Bimpl} \top (B' \cup \{\alpha\})$*. Then* $(B' \cup \{\alpha\} \cup \mathit{Bim}_i) \equiv_{\mathcal{P}} \lceil B \rceil_i \cup \{\alpha\}$*.*

Based on this lemma, one can show that choice revision with a DS-selection function c such that $c(I) = i$ is representable by $\lceil B \rceil_i \wedge \alpha$, and skeptical revision is representable as $\bigvee_{i \in I} \lceil B \rceil_i \wedge \alpha$.

We define an adapted flipping operator, called polarity flipping. For ease of definition we assume that only the connectors $\wedge, \vee$ and $\neg$ appear in the knowledge base (otherwise just recompile the KB equivalently). An occurrence of a propositional symbol is *syntactically positive* iff it occurs in the scope of an even number of negation symbols, otherwise it is *syntactically negative*. Let $(\mathit{Im}_i)_{i \in I}$ be the family of belief extensions for B and α, and let Im_k be an implication-based belief extension chosen by the selection function, $c(I) = k$. The result of *partial flipping* to B, for short $\lceil B \rceil_k^{\rightarrow}$, is defined as follows: If $p \rightarrow p' \notin \mathit{Im}_k$, then switch the polarity of the negative occurrences of p in $\bigwedge B$ (by adding $\neg$ in front of these occurrences). If $p' \rightarrow p \notin \mathit{Im}_k$, then switch the polarity of the positive occurrences of p in $\bigwedge B$. Let $\lceil B \rceil^{\rightarrow} = \bigvee_{i \in I} \lceil B \rceil_i^{\rightarrow}$.

With these definitions one can show the finite representability.

Theorem 9. $B *_{DS}^{c,\rightarrow} \alpha$ *has the same models as* $\lceil B \rceil_c^{\rightarrow} \wedge \alpha$ *and that* $B *_{DS}^{\rightarrow} \alpha$ *has the same models as* $\lceil B \rceil^{\rightarrow} \wedge \alpha$*.*

For the proof of 9 and also for other theorems below we introduce the forgetting operator. Let Θ_S denote an operator that, given a formula α and a set S of symbols $S \subseteq \mathcal{P}$, computes a formula representing all consequences of α that do not contain symbols in S, i.e., Θ_S forgets about S. For $\mathcal{I} \in \mathrm{Int}(S)$ let $\alpha_{\mathcal{I}}$ be defined as follows: Substitute all occurrences of $p \in S$ in α where $p^{\mathcal{I}} = \mathcal{I}(p) = 1$ by $\top$, else $\bot$ is substituted for p. Now one can define $\Theta_S : \alpha \mapsto \bigvee_{\mathcal{I} \in \mathrm{Int}(S)} \alpha_{\mathcal{I}}$. For arbitrary $S \subseteq \mathcal{P}$ let $\Theta_S(\alpha) = \Theta_{\mathrm{symb}(\alpha) \cap S}(\alpha)$. For example, let $\alpha = (p \wedge q) \vee (r \wedge s)$ and $S = \{p, r\}$. Then $\Theta_S = ((\bot \wedge q) \vee (\bot \wedge s)) \vee ((\bot \wedge q) \vee (\top \wedge s)) \vee ((\top \wedge q) \vee (\bot \wedge s)) \vee ((\top \wedge q) \vee (\top \wedge s))$. This is equivalent to the formula $s \vee q$. The following facts concerning Θ_S for $S \subseteq \mathcal{P}$ can be proved easily. For all $\alpha \in \mathrm{Fml}(\mathcal{P})$: $\alpha \models \Theta_S(\alpha)$ and $\mathrm{Cn}^{\mathcal{P} \setminus S}(\alpha) = \mathrm{Cn}^{\mathcal{P} \setminus S}(\Theta_S(\alpha))$. Note, that $\Theta_S(\alpha)$ can be described as the quantified boolean formula $\exists S.\alpha$.

case	form in $\lceil \mathcal{B} \rceil_k^{\rightarrow}$	form in B	implications in Impl_k
I	p	p	$p' \rightarrow p, p \rightarrow p'$
II	p	p	$p' \rightarrow p$
III	p	$\neg p$	$p' \rightarrow p$
IV	$\neg p$	$\neg p$	$p' \rightarrow p, p \rightarrow p'$
V	$\neg p$	$\neg p$	$p \rightarrow p'$
VI	$\neg p$	p	$p \rightarrow p'$

Table 1: Cases for literals

Proof of Theorem 9

Let $(\mathit{Im}_i)_{i \in I}$ be the set of all implication based consistent belief set extensions for B and $\{\alpha\}$. First note that the maximality of the Im_i has the effect that for every $p \in \mathcal{P}$ at least one of $p \rightarrow p', p' \rightarrow p$ is contained in Im_i. Because, suppose that neither of $\overrightarrow{p}, \overleftarrow{p}$ is contained in Im_i. The maximality of Im_i implies that $B' \cup \mathit{Impl}_i \cup \{\alpha\} \models \neg \overrightarrow{p} \wedge \neg \overleftarrow{p}$ and so $B' \cup \mathit{Im}_i \cup \{\alpha\} \models \bot$, which contradicts the fact that $B' \cup \mathit{Im}_i \cup \{\alpha\}$ is consistent.

Now we start the proof of the theorem by assuming that that B is a formula in DNF. Let $c(I) = k$. We show that $B' \cup \mathit{Im}_k \cup \{\alpha\} \equiv_{\mathcal{P}} \lceil \mathcal{B} \rceil_k^{\rightarrow} \cup \{\alpha\}$ by proving the two implicit directions.

'Right to left': Let $B' \cup \mathit{Im}_k \cup \{\alpha\} \models \beta$ for $\beta \in \mathrm{Fml}(\mathcal{P})$. We have to show $\lceil \mathcal{B} \rceil_k^{\rightarrow} \cup \{\alpha\} \models \beta$. Let be given a model $\mathcal{I} \models \lceil \mathcal{B} \rceil_k^{\rightarrow} \cup \{\alpha\}$. Then there is a dual clause cl in $\lceil \mathcal{B} \rceil_k^{\rightarrow}$ such that $\mathcal{I} \models cl$. (Remember: that a dual clause is just a conjunction of literals). For every literal li in cl one of the cases mentioned in Table 1 holds.

So there are 6 different types of literals in cl; this justifies the following representation of cl in $\lceil \mathcal{B} \rceil_k^{\rightarrow}$.

$$\begin{aligned} kl \quad = \quad & p_1^1 \wedge \cdots \wedge p_{n_1}^1 \wedge p_1^2 \wedge \cdots \wedge p_{n_2}^2 \wedge p_1^3 \wedge \cdots \wedge p_{n_3}^3 \\ & \wedge \neg p_1^4 \wedge \cdots \wedge \neg p_{n_4}^4 \wedge \neg p_1^5 \wedge \cdots \wedge \neg p_{n_5}^5 \\ & \wedge \neg p_1^6 \wedge \cdots \wedge \neg p_{n_6}^6 \end{aligned}$$

Define a new interpretation $\mathcal{I}'$ in the following way:

- $\mathcal{I}'(p_i'^1) = \mathcal{I}'(p_i^1) = 1 = \mathcal{I}(p_i^1)$;
- $\mathcal{I}'(p_i'^2) = \mathcal{I}'(p_i^2) = 1 = \mathcal{I}(p_i^2)$;
- $\mathcal{I}'(p_i'^3) = 0 \neq \mathcal{I}(p_i^3) = 1; \mathcal{I}'(p_i^3) = \mathcal{I}(p_i^3) = 1$;
- $\mathcal{I}'(p_i'^4) = \mathcal{I}'(p_i^4) = 0 = \mathcal{I}(p_i^4)$;
- $\mathcal{I}'(p_i'^5) = \mathcal{I}'(p_i^5) = 0 = \mathcal{I}(p_i^5)$;
- $\mathcal{I}'(p_i'^6) = 1 \neq \mathcal{I}(p_i^6) = 0; \mathcal{I}'(p_i^6) = \mathcal{I}(p_i^6) = 0$;
- if r is a propositional symbol in $\mathcal{P}$ with $r \neq p_i^j$ and $r' \neq p_i'^j$, let $\mathcal{I}'(r') = \mathcal{I}(r)$;

From the construction of $\mathcal{I}$ it follows that $\mathcal{I}'_{\restriction \mathcal{P}} = \mathcal{I}_{\restriction \mathcal{P}}$ and $\mathcal{I}' \models B' \cup \mathit{Im}_k \cup \{\alpha\}$. So $\mathcal{I}' \models \beta$ and hence $\mathcal{I} \models \beta$.

'Left to right': Now suppose that $\lceil \mathcal{B} \rceil_k^{\rightarrow} \models \beta$ and let $\mathcal{I} \models B' \cup \mathit{Im}_k \cup \{\alpha\}$. That is, there is a dual clause cl' in B' of the form

$$p_1'^1 \wedge \cdots \wedge p_{n_1}'^1 \wedge p_1'^2 \wedge \cdots \wedge p_{n_2}'^2 \wedge \neg p_1'^3 \wedge \cdots \wedge \neg p_{n_3}'^3$$
$$\wedge \neg p_1'^4 \wedge \cdots \wedge \neg p_{n_4}'^4 \wedge \neg p_1'^5 \wedge \cdots \wedge \neg p_{n_5}'^5$$
$$\wedge p_1'^6 \wedge \cdots \wedge p_{n_6}'^6$$

It follows that $\mathcal{I}(p_i^1) = \mathcal{I}(p_i^2) = 1$ and $\mathcal{I}(p_i^2) = \mathcal{I}(p^5) = 0$. (Because of the types of the literals and the fact that the hypotheses are made true.) Moreover, as $p_i^3 \rightarrow p_i'^3$ and $p_i'^6 \rightarrow p_i^6$ are not in Im_k, the maximality of Im_k entails $B' \cup \mathit{Im}_k \cup \{\alpha\} \models p_i^3 \wedge \neg p_i'^3$ and $B' \cup \mathit{Im}_k \cup \{\alpha\} \models \neg p_i^6 \wedge \neg p_i'^6$. Therefore we also have $\mathcal{I}(p_i^3) = 1$ and $\mathcal{I}(p_i^6) = 0$. Finally, this entails $\mathcal{I} \models \lceil \mathcal{B} \rceil_k^{\rightarrow} \wedge \alpha$, hence $\mathcal{I} \models \beta$.

Proof of Theorem 2

With Theorem 9 Theorem 2 is an immediate corollary: Assume that b is given in complete disjunctive normal form. $\mathcal{I} \models \lceil B \rceil^{\rightarrow} \wedge \alpha$ iff $\mathcal{I} \models \lceil B \rceil_i^{\rightarrow}$ for some i. Now, in $\lceil B \rceil_i^{\rightarrow}$ all propositional variables p for which either $\overrightarrow{p}$ or $\overleftarrow{p}$ does not occur in Im_i all associated occurrences in the dual clauses (which correspond actually to models) are flipped into the other polarity so that all dual clauses have for each occurrence of p the same polarity. But this means that there is a model $\mathcal{J} \models B$ which differs from $\mathcal{I}$ exactly in the corresponding polarities for the ps with missing bridging axioms in Im_i; this is the same as saying that $\mathcal{I} \Delta_{\pm} \mathcal{J} \in \Delta_{\pm}^{min}(B, \alpha)\mathcal{J}$.

Proof of Theorem 3

We define γ as follows:

$$\gamma(H) \quad = \quad \{ X \in H \mid X \cap \mathit{Bimpl} \text{ is maximal in } \{X' \cap \mathit{Bimpl} \mid X' \in H\} \}$$

γ selects from H those sets for which the intersection with the set of bi-implications *Bimpl* is maximal. Note, that this definition is completely independent of B, and hence the content of the theorem is stronger than to say that for any B one my define a selection function γ such that the representation holds. Let $(\mathit{Bim}_j^{\vee})_{j \in J}$ be the family of sets in $\gamma(\overline{\mathit{Bimpl}} \top (B' \cup \{\alpha\}))$. Because of the definition of the disjunctive closure and of the remainders it holds that for all $i \in I$ there is a $j \in J$ s.t. $\mathit{Bim}_i \subseteq \mathit{Bim}_j^{\vee}, \mathit{Bim}_j^{\vee} \cap \mathit{Bimpl} = \mathit{Bim}_i$ and

$$\mathit{Bim}_j^{\vee} \subseteq \mathrm{Cn}(\mathit{Bim}_i) \tag{1}$$

Conversely, because of the definition of γ one has for every $j \in J$ an $i \in I$ such that

$$\mathit{Bim}_j^{\vee} \supseteq \mathit{Bim}_i \tag{2}$$

Proof of $B *_{DS} \alpha \supseteq \mathrm{Cn}^{\mathcal{P}}(\overline{\mathit{Bimpl}}) *_{\gamma} (B' \cup \{\alpha\}))$*:* Assume first that $\beta \in \mathrm{Cn}^{\mathcal{P}}(\overline{\mathit{Bimpl}} *_{\gamma} (B' \cup \{\alpha\}))$, i.e., $\beta \in \mathrm{Fml}(\mathcal{P})$ and $(\bigcap_{j \in J} \mathit{Bim}_j^{\vee}) \cup B' \cup \{\alpha\} \models \beta$. So, for all $j \in J$ it holds that $\mathit{Bim}_i^{\vee} \cup B' \cup \{\alpha\} \models \beta$ and so $\mathit{Bim}_j^{\vee} \models (\bigwedge B' \wedge \alpha) \rightarrow \beta$. Together with (1) it follows that for all $i \in I$: $\mathit{Bim}_i \models (\bigwedge B' \wedge \alpha) \rightarrow \beta$, hence $(\bigwedge B' \wedge \alpha) \rightarrow \beta \in \mathrm{Cn}(\mathit{Bim}_i) \subseteq \mathrm{Cn}(\mathit{Bim}_i \cup B' \cup \{\alpha\}) = E_i$ for all $i \in I$. Consequently, $\beta \in E_i$ for all $i \in I$ and lastly $\beta \in \bigcap_{i \in I} E_i = B *_{\mathrm{DS}} \alpha$.

Proof of $B *_{DS} \alpha \subseteq \mathrm{Cn}^{\mathcal{P}}(\overline{\mathit{Bimpl}} *_{\gamma} (B' \cup \{\alpha\}))$*:* Let $\beta \in B *_{\mathrm{DS}} \alpha = \bigcap_{i \in I} E_i$, i.e. $\beta \in \mathrm{Fml}(\mathcal{P})$, and for all $i \in I$: $\mathit{Bim}_i \cup B' \cup \{\alpha\} \models \beta$ and hence $\mathit{Bim}_i \models (\bigwedge B' \wedge \alpha) \rightarrow \beta$. Because of the compactness property of propositional logic one has for every $i \in I$ a finite subset $\mathit{Bim}_i^f \subseteq \mathit{Bim}_i$ such that $\mathit{Bim}_i^f \models (\bigwedge B' \wedge \alpha) \rightarrow \beta$. Because B is finite, so is the set I, which is the index set of all belief extensions E_i. Let $I = \{1, \ldots, k\}$. There are only finitely many maximal sets of bridging axioms Bim_i and finitely many extensions E_i. So the disjunction $\bigvee_{i \in I} \mathit{Bim}_i^f$ is defined and the following holds :

$$\bigvee_{i \in I} \mathit{Bim}_i^f \models (\bigwedge B' \wedge \alpha) \rightarrow \beta \tag{3}$$

For all $i \in I$ let $n_i = |\mathit{Bim}_i|$ be the number of elements in Bim_i^f and $N_i = \{1, \ldots, n_i\}$. Every set Bim_i, $i \in I$, is representable as $\mathit{Bim}_i \equiv \bigwedge_{j=1}^{n_i} (p_{ij} \leftrightarrow p'_{ij})$. Applying the distribution law $\bigvee_{i \in I} \mathit{Bim}_i^f$ is transformable in a conjunction of disjunctions of bi-implications:

$$\bigvee_{i \in I} \mathit{Bim}_i^f \equiv \bigwedge_{(j_1, \ldots, j_k) \in N_1 \times \cdots \times N_k} \bigvee_{i}^{k} (p_{j_i} \leftrightarrow p'_{j_i}) \tag{4}$$

Because of (2) for every $j \in J$ there is an $i \in I$ with $\mathit{Bim}_j^{\vee} \supseteq \mathit{Bim}_i$. Now for every $(j_1, \ldots, j_k) \in N_1 \times \cdots \times N_k$ it holds that $(p_{j_i} \leftrightarrow p'_{j_i}) \in \mathit{Bim}_i$ and hence for every $(j_1, \ldots, j_k) \in N_1 \times \cdots \times N_k$ also $\bigvee_i^k (p_{j_i} \leftrightarrow p'_{j_i}) \in \mathit{Bim}_j^{\vee}$ holds. Hence for every $j \in J$ it is true that $\mathit{Bim}_j^{\vee} \models \bigwedge_{(j_1, \ldots, j_k) \in N_1 \times \cdots \times N_k} \bigvee_i^k (p_{j_i} \leftrightarrow p'_{j_i})$. With (4) it follows that for every $j \in J$ that $\mathit{Bim}_j^{\vee} \models \bigvee_{i \in I} \mathit{Bim}_i^f$ and with the entailment in (3) it further follows that $\mathit{Bim}_j^{\vee} \models (\bigwedge B' \wedge \alpha) \rightarrow \beta$. In the end: $\beta \in \mathrm{Cn}^{\mathcal{P}}(\overline{\mathit{Bimpl}} *_{\gamma} (B' \cup \{\alpha\}))$.

Proof of Theorem 4

Follows directly from Thm. 3 and Thm. 1

Proof of Theorem 5

The proof for the equivalent representation of $*_{\mathrm{DS}}^{\rightarrow}$ by an disjunctively closed implication-based reinterpretation operator uses the same construction as in the proof of Theorem 3, as the construction does not use the special property of biimplications. With this also the represntation for weak Satoh follows due to Thm. 2.

Proof of Theorem 6

For the proof of this theorem we use an alternative characterization of Weber revision with the forgetting operator Θ_S, namely,

$$[\![B *_W \alpha]\!] = [\![\Theta_{\Omega(B,\alpha)}(B) \wedge \alpha]\!]$$

According to definition $B \circ^{\mathit{Bimpl}} \alpha = \underbrace{\bigcap (\mathit{Bimpl} \top (B' \cup \alpha))}_{=:X} \cup B' \cup \{\alpha\}$. Due to interpolation we have $\mathrm{Cn}^{\mathcal{P}}(X \cup B' \cup \alpha) = \mathrm{Cn}^{\mathcal{P}}(\mathrm{Cn}^{\mathcal{P}}(X \cup B') \cup \alpha)$. Now one can verify that

$$\mathrm{Cn}^{\mathcal{P}}(X \cup B') = \mathrm{Cn}^{\mathcal{P}}(\Theta_{\{p \in \mathcal{P} \overleftrightarrow{p} \notin X\}}(B))$$

(Because, for example, $\Theta_{p',q'}(B' \cup p' \leftrightarrow p)) = (B'[p'/1, q'/0] \wedge p) \vee (B'[p'/1, q'/1] \wedge p) \vee (B'[p'/0, q'/0] \wedge \neg p) \vee (B'[p'/0, q'/1] \wedge \neg p) \equiv_{\mathcal{P}} (B'[p'/p, q'/0] \vee B'[p'/p, q'/1] = \Theta_{\{q\}}(B).)$ Now $\{p \in \mathcal{P} \mid \overleftrightarrow{p} \notin X\} = \Omega(B, \alpha)$. Hence due to $[\![B *_W \alpha]\!] = [\![\Theta_{\Omega(B,\alpha)}(B) \cup \{\alpha\}]\!]$, the assertion follows.

Proof of Theorem 7

The proof works in the same way as the proof of Thm. 6. For this we use the alternative characterization of weak Weber as

$$[\![B *_W \alpha]\!] = [\![\Theta_{pr_1(\Omega_\pm(B,\alpha)) \cup pr_2(\Omega_\pm(B,\alpha))}(B) \wedge \alpha]\!]$$

Proof of Theorem 8

We give here only the proof idea which relies in the correct definition of the selection function γ. Let $\Psi := \{p', \neg p' \mid p \in \mathcal{P}\}$.

$$\begin{aligned}\gamma(Z) \quad = \quad & \{X \in Z \mid X \cap \Psi \text{ is inclusion maximal within} \\ & \text{all } X' \cap \Psi \in Z \text{ and } X \cap \textit{Bimpl} \\ & \text{is inclusion maximal within all} \\ & X'' \cap \textit{Bimpl} \text{ for which there is } X''' \subseteq \textit{Bimpl} \\ & \text{with } X'' = (X \cap \Psi) \cup X'''\}\end{aligned}$$

This selection function first selects maximal sets of primed literals from Ψ. These maximal sets correspond just to the the models of B'. Then it chooses maximal sets of the disjunctive closure of the bi-implications. But as was shown for the representation of Satoh revision by disjunctively closed bi-implications, this corresponds to considering minimal symmetrical difference.