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A Representation Theorem for Spatial Relations

Özgür L. Özçep

Institute of Information Systems (IFIS)
University of Lübeck
Lübeck, Germany
oezcep@ifis.uni-luebeck.de

Abstract. Spatial relations have been investigated in various inter-related areas such as qualitative spatial reasoning (for agents moving in an environment), geographic information science, general topology, and others. Most of the results are specific constructions of spatial relations that fulfill some required properties. Results on setting up axioms that capture the desired properties of the relations are rare. And results that characterize spatial relations in the sense that they give a complete set of axioms for the intended spatial relations still have to be presented. This paper aims at filling the gap by providing a representation theorem: It shows that there is a finite set of axioms that are fulfilled by a binary relation if and only if it can be constructed as a binary spatial relation based on a nested partition chain.

Keywords: spatial relation, axiomatization, representation

1 Introduction

Spatial relations have been investigated in various inter-related areas such as qualitative spatial reasoning [16], geographic information science [17], general topology [9], and others. Most of the results achieved are specific constructions of spatial relations that fulfill some desired properties—which may vary according to the application/modeling context. Although the axiomatic method is a well-proven approach for the description of entities, results on setting up axioms that capture the desired properties of spatial relations are rare. And results that characterize spatial relations in the sense that they give a complete set of axioms for the intended spatial relations still have to be achieved.

The present paper aims at filling this gap with a semantic analysis of a specific class of spatial relations: Not only does it set up an axiom set that the intended spatial relation should fulfill but it takes a deeper look into the structure of the models for the axioms. The general idea is to systematically characterize the models by grouping them into disjoint, mathematically well-defined classes. A particularly interesting case is the one in which the set of models is described by exactly one class of models built according to some construction principle. In this case, the axioms really characterize the intended concepts, providing a canonical representation according to the construction principle of the class. A well-known application of this methodology is Stones representation theorem [15]. A nice side-product of a representation theorem is that unintended models, which could result from an incomplete axiomatization, are excluded.

The spatial relations for which this paper gives a representation theorem are defined on the basis of a special structure, a hierarchal structure of nested partitions [5,10,11]. Typical examples of such total orders of nested partitions are made up of administrative units where the administrative units in a rougher granularity (e.g., districts) are the unions of administrative units of the lower level (e.g., municipalities). For example, think of two partitions of Switzerland, where the first partition consists of municipalities and where the second consists of districts. All districts are municipalities or are unions of two or more municipalities.

The interest in such types of spatial relations stems from observations regarding the context-dependency of spatially relatedness: The criterion for deciding whether a is considered spatially related to b depends on the type of the object that has b as its spatial extension. If b is a natural object such as a mountain, then the spatial criteria (be it geometric, topological, or metric) for regarding objects spatially related may depend on scaling contexts of (big) natural borders such as those of forests or rivers etc. But if the object with spatial extension b is a non-natural artifact such as a house, then different criteria have to be taken account: say borders made up by cadastral data.

The situation, as the house example demonstrates, may even be more complicated: it may be the case that the same spatial area b (house area) is the spatial extension of two different objects: the house considered as the pure geometrical object or the house considered as a legal object which has to adhere to planning laws. Depending on which objects are relevant for the use case different criteria are relevant in order to decide whether an object is spatially related to a house: In the first case, a purely metric criterion is in order, in the second case the district or even the country in which the house is situated is in order.

In this paper, only objects of the same type are considered. (With respect to the example above this means: we either consider only houses as pure geometrical objects or consider only houses as legal objects.) Hence, for spatially relatedness there is one context criterion fixed according to which two objects are considered to be related or not. This criterion is formalized by nested partitions of a spatial domain X. A partition provides a granularity or scale w.r.t. which the spatial relatedness of two regions is fixed; the main idea is to consider one of the arguments (here the second one) as the one determining the scaling context, i.e., the level on the ground of which two regions are defined to be spatially related or not. The results of this paper can be easily generalized to the case of regions with different types by considering collections of nested partition chains.

With this model in mind, the representation theorem now reads as follows: There is a finite set of axioms such that any binary relation fulfilling these can be represented as a spatial relation based on a nested partition chain.

The rest of the paper is structured as follows. Section 2 recapitulates the definitions of partition chains and spatially relatedness. Section 3 gives a comparison of spatially relatedness with proximity. Section 4 defines the upshift operator used in the axioms. Section 5 contains the main axioms for the representation theorem, which is proved in Section 6. The last two sections deal with related work and give a conclusion.[1]

2 Partitions and Spatially Relatedness

This work builds on the nearness framework developed by [11], which in turn is an abstraction of the framework by [10]—the abstraction being the transition from regions [12] to arbitrary sets. Following [11], it is assumed in this paper that X (the domain of objects which are the candidates for extensions of regions) is just a finite set. Hence, the results of this paper are relevant for discrete/digital topology [4,13].

We recapitulate the main technical concepts of a partition and of a normal partition chain. The usual partition concept of set theory will be called *set partition*. That is, given X and a family of sets $\{a_i\}_{i\in I}$ where the a_is are pairwise disjoint is a set partition iff X is the union of all the a_is, formally $X = \biguplus_{i\in I} a_i$.

Definition 1 (partition). *A* partition *of a set X on level $i \in \mathbb{N}$ is a family of pairs $(i, a_j)_{j\in J}$ s.t. $(a_j)_{j\in J}$ is a set partition of X. A pair $c = (i, a_j)$ is called a* cell of level i*. Its underlying set a_j (the second argument) is denoted $us(c)$.*

Partition chains are partitions of X that are nested.

Definition 2 (partition chain). *Consider a collection of $n+1$ different partitions of X where all partitions have only finitely many cells. This set of partitions is called a* partition chain *pc iff*

1. *all cells $(i+1, a_j)$ of level $i+1$ (for $i \in \{0, \ldots, n-1\}$) are unions of i-level cells, i.e., there exist (i, b_k), $k \in K$, such that $a_j = \biguplus_{k\in K} b_k$;*
2. *and the last partition (level n) is made up by (X).*

Every cell has a unique upper cell. For a cell (i, a_j) (with $1 \leq i \leq n-1$) let $(i, a_j)^{\uparrow,pc} = (i+1, a_k)$ be the unique cell of the upper level in this partition chain pc such that $a_j \subseteq a_k$. For the cell of level n set $(n, X)^{\uparrow,pc} = (n, X)$. The cell $(i, a_j)^{\uparrow,pc}$ is called the upper cell *of (i, a_j). If the partition chain is clear from the context, let $(i, a_j)^{\uparrow}$ stand for $(i, a_j)^{\uparrow,pc}$.*

A partition chain is normal *iff all set partitions underlying the partitions are pairwise distinct. A partition chain is* strict *iff for every level i, $i > 0$ and every cell (i, a_j) there is no cell on the level below with the same underlying set $(i-1, a_j)$.*

Example 1. An example of a (strict) partition chain with three levels is illustrated in Figure 1, where we give a region oriented presentation (left) and the tree structure (right) with the associated levels. In order to make the example fully concrete we assume that $us(X) = \{1, 2, 3, 4, 5, 6\}$ and $c_i = (0, \{i\})$, for $i \in us(X)$.

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[1] This is the extended version of a paper submitted to AI 2015.
It contains an appendix with additional proofs.

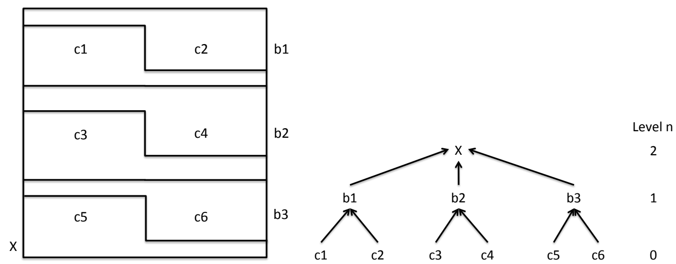


Fig. 1. A strict partition chain $(c_i)_{i\in\{1,2,3,4,5,6\}} \leq (b_i)_{i\in\{1,2,3\}} \leq (X)$

For a subset $b \neq \emptyset$ of X let $\tilde{b}^{pc}$ denote the cell (i, a_j) such that $b \subseteq a_j$ and i is minimal. The integer $i = l_{pc}(b)$ is called the level of b in pc. If the partition chain pc is unique in the used context, it is not mentioned in the subscripts. As a shorthand for $(\tilde{b}^{pc})^{\uparrow,pc}$ one may write $b^{\Uparrow,pc}$.

Example 2. Consider again Figure 1. For the set $\{5,6\} = us(c_5) \cup us(c_6)$ we have $\widetilde{\{5,6\}} = b_3$ and so $\{5,6\}^{\Uparrow,pc} = X$. For the set $\{3,6\} = us(c_3) \cup us(c_6)$ we already have $\widetilde{\{3,6\}} = X$, and so again $\{3,6\}^{\Uparrow,pc} = X$.

Definition 3 (spatially relatedness sr). *For a normal partition chain pc over X spatially relatedness sr_{pc} is defined by:*

$$\mathsf{sr}_{pc}(a,b) \text{ iff } a \cap us(b^{\Uparrow,pc}) \neq \emptyset \tag{1}$$

So the main idea of the spatial relation is that the second argument (here b) determines the partition level w.r.t. which the first argument (here a) is considered to be related; if b is a cell, then one checks whether the intersection of a with the upper cell of b is non-empty. If it is non-empty, then a is spatially related to b. Otherwise a is not spatially related to b. If b is not the underlying set of a cell, then one looks for the smallest upper cell whose underlying set contains b and then proceeds as before.

Example 3. We consider the partition chain in Figure 2. It is similar to that of Figure 1, but here let $X = \{1,2,\ldots,7,8\}$, $c_i = (0,i)$ for $i \in \{1,2,3,4\}$ and $c_5 = (0,\{5,7\})$, $c_6 = (0,\{6,8\})$. Moreover, there is a set (region) $z = \{7,8\}$ which overlaps with the cells c_5 and c_6 and a set $w = \{6\}$ contained in the cell c_6. We have $\tilde{z} = b_3 = (1,\{5,6,7,8\})$ and hence $z^{\Uparrow,pc} = X$. So every set $b \subseteq us(X)$ is spatially related to z, i.e., $\mathsf{sr}_{pc}(b,z)$. In contrast, consider the set w. Here one has $\tilde{w} = c_6 = (0,\{6,8\})$ and hence $w^{\Uparrow,pc} = b_3 = (1,\{5,6,7,8\})$. So only sets not disjoint from $\{5,6,7,8\}$ are near w.

Example 4. This example illustrates the difference between (metrical) nearness and spatial relatedness. Assume that we consider cadastral data covering two different nations and we consider houses as legal objects adhering to planning laws. Two houses a and b are sited on different sides of the border line of two

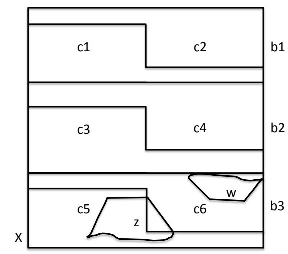


Fig. 2. Partition Chain with Non-Cells for Illustrating Spatially Relatedness

nations that have two completely different planning laws. Then, the legal object a would not stand in sr relation to the legal object b (w.r.t. the partition made of the cadastral data) though the areas a and b are clearly metrically near. Because of this we use the more neutral term *spatially relatedness* instead of *nearness* as used by [10].

3 Spatially Relatedness vs. Proximity

The partition based spatial relations have some connections to but nonetheless are different from minimal proximity relations δ. Structures (X, δ) with domain X and a binary relation δ over X are *minimal proximity structures* [4] iff the following axioms are fulfilled (where $a, b, c \subseteq X$):

(P1) If $\delta(a, b)$, then a and b are nonempty.
(P2*) $\delta(a, b)$ or $\delta(a, c)$ iff $\delta(a, (b \cup c))$.
(P3) $\delta(a, c)$ or $\delta(b, c)$ iff $\delta((a \cup b), c)$.

Obviously, sr_{pc} fulfills (P1) and (P3), but only the following weakening of (P2*):

(P2) If $\delta(a, b)$ or $\delta(a, c)$, then $\delta(a, (b \cup c))$

Moreover one can show that sr_{pc} fulfills the following two properties:

(P4) If $a \cap b \neq \emptyset$, then $\delta(a, b)$ and $\delta(b, a)$.
(P5) For all $a \subsetneq X$ with $a \neq \emptyset$: $\delta(a, (X \setminus a))$ or $\delta((X \setminus a), a)$.

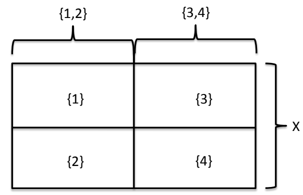
The following proposition [10] summarizes these results.

Proposition 1. *All* sr_{pc} *for normal partition chains pc fulfill the axioms (P1), (P2), (P3), (P4), and (P5).*

As the following example shows, this set of axioms is incomplete in the following sense: There are still models where δ is interpreted by a binary relation that cannot be represented as sr_{pc} for an appropriate partition chain pc; in other words, these axioms do not completely characterize/represent the relations of the type sr_{pc}.

Example 5. Assume $X = \{1, 2, 3, 4\}$ and the following δ-relations are given:

- None of the following holds: $\delta(\{4\}, \{2\})$, $\delta(\{4\}, \{1\})$, $\delta(\{1\}, \{3\})$, $\delta(\{1\}, \{4\})$
- for all other $a, b \subseteq X$ with $a, b \neq \emptyset$ it holds that $\delta(a, b)$.



It can be easily checked that δ fulfills (P1)–(P5), but that it is not representable as sr_{pc} for a normal partition chain.

The last assertion is proved as follows: Take the assertion $\delta(3, 2)$. Assume that there is a normal pc such that $\delta = \mathsf{sr}_{pc}$. Consider the following cases:

1. $c := \widetilde{\{2\}} = (0, \{2\})$. As $\delta(\{1\}, \{2\})$ and $\delta(\{3\}, \{2\})$ we must have $\{1, 2, 3\} \subseteq us(c^{\uparrow})$. As not $\delta(\{4\}, \{2\})$, $us(c^{\uparrow}) = \{1, 2, 3\}$. That means that on level 1 one can have only the sets $\{1, 2, 3\}$ and $\{4\}$ as underlying cells. But this means that $\widetilde{\{4\}} = (0, \{4\})$ and $4^{\Uparrow} = (1, \{4\})$. But this contradicts the fact that $\delta(\{2\}, \{4\})$ holds while one would have to have not $\mathsf{sr}_{pc}(2, 4)$.
2. In the other cases $c := \widetilde{\{2\}} = (0, a)$ for a set a with $\{2\} \subsetneq a$. But then $c^{\uparrow} = (1, b)$ for a set b which must again be $b = \{1, 2, 3\}$ for the same reasons as in the former case. But then one gets a contradiction again.

4 The Upshift Operator

The main idea for the representation theorem is to reconstruct the levels by referring only to δ. A first step towards this end is to define the *upshift operator* $\cdot^{\Uparrow_\delta}$, an abstract analogue of the level shifting operator $\cdot^{\Uparrow,pc}$. The upshift operator is going to be defined below as a unique function based on δ. In all axioms where $\cdot^{\Uparrow_\delta}$ occurs it can be unfolded to its defining formula to get rid of the new symbol.

Given δ, the equivalence relation ${}^{\bullet}\!\!\sim$ is defined as follows:

$$a \,{}^{\bullet}\!\!\sim b \text{ iff } \{c \subseteq X \ \mid \delta(c, a)\} = \{c \subseteq X \ \mid \delta(c, b)\} \tag{2}$$

This equivalence relation can be formulated for any relation δ, independently of the specific properties of δ. As usual, for any equivalence relation $\sim$, $[a]_\sim$ denotes the equivalence class of a w.r.t. $\sim$. A simple observation is the following:

Proposition 2. *For partition chains pc and $a, b \subseteq X$ s.t. $\tilde{a} = (i, a)$, $\tilde{b} = (i, b)$, and $a^{\Uparrow,pc} = b^{\Uparrow,pc}$ it holds that $a \,{}^{\bullet}\!\!\sim b$.*

Definition 4. *Given a binary relation δ, the* upshift operator $\cdot^{\Uparrow_\delta}$ for δ *is defined for any nonempty set $b \subseteq X$ as follows:*

$$b^{\Uparrow_\delta} = \bigcup [b]_{{}^{\bullet}\!\sim} \tag{3}$$

So the set $b^{\Uparrow_\delta}$ is just the union of all sets a that have the same set of sets δ related it as b. If a partition chain pc over X is given, then one has two different shift operators, the operator $\cdot^{\Uparrow,pc}$, which calculates the upper cell w.r.t. pc, and the sr_{pc} level shift operator $\cdot^{\Uparrow_{\mathsf{sr}_{pc}}}$. As the following proposition shows, the δ-shift operator is nothing else than the level shifting operator in case of $\delta = \mathsf{sr}_{pc}$.

Proposition 3. *Let pc be a partition chain over X. Then for any nonempty* $b \subseteq X$ *the following equality holds:* $b^{\Uparrow,pc} = b^{\Uparrow_{\mathsf{sr}_{pc}}}$.

5 The Main Axioms

In the following subsections the main axioms are introduced that make up the representation theorem.

5.1 Spatially Relatedness is Grounded

The following axiom states a necessary and sufficient condition for the spatially relatedness of two sets with reference to the upshift operator $\cdot^{\Uparrow_\delta}$. It says that a δ-related to b if and only if a has a non-empty intersection with the upshift of b.

(Pgrel) For all $a, b \subseteq X$: $\delta(a, b)$ iff $a \cap b^{\Uparrow_\delta} \neq \emptyset$.

The axiom expresses a principle on the connection between the abstract δ relation and the set-theoretic element-of relation $\in$: Namely that δ is **gr**ounded in the **el**ement relation (hence the acronym grel).

Unfolding (Pgrel) w.r.t. the definition of $\cdot^{\Uparrow_\delta}$ results in the axiom (Pgel'):

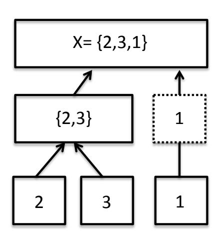
(Pgrel') For all $a, b \subseteq X$: $\delta(a, b)$ iff there is some c such that $c \overset{\bullet}{\sim} b$ and $a \cap c \neq \emptyset$.

Further, the relation symbol $\overset{\bullet}{\sim}$ can be eliminated—leading to (Pgrel") which refers only to δ (and some set operations).

(Pgrel") For all $a, b \subseteq X$: $\delta(a, b)$ iff there is some c such that for all z: $\delta(z, c)$ iff $\delta(z, b)$, and $a \cap c \neq \emptyset$.

Intuitively speaking, (Pgrel") says that a is δ-related to b iff it has a non-empty intersection with a set c that is similar ($\overset{\bullet}{\sim}$ equivalent to) to b. Looking at the unfolding, it is no surprise that (Pgrel) on its own is not expressive enough to characterize sr_{pc} and hence is far away from being a definition of sr_{pc}. This is demonstrated by Example 6.

Example 6. Let $X = \{1, 2, 3\}$ and δ be as follows:
- for all $a \subseteq X$: $\delta(a, \{1\})$
- $\neg\delta(1, 2)$
- $\neg\delta(1, 3)$



One calculates $2^{\Uparrow_\delta} = 3^{\Uparrow_\delta} = \{2, 3\}$ and $\{2, 3\}^{\Uparrow_\delta} = X$ and shows that (Pgrel) is fulfilled. Nonetheless this δ is not representable as sr_{pc} for some normal pc.

The reason for non-representability in the above example is that there is no appropriate level notion. All the sets $\{1\}, \{2\}$, and $\{3\}$ would have to be of level 0. Applying $\cdot^{\Uparrow\delta}$ to $\{2\}$ and $\{3\}$ gives $\{2,3\}$, but the application to $\{1\}$ already gives X. Hence $\{1\}$ would have to appear on two levels (serving also as a cell on the level of $\{2,3\}$), but this would mean that the only set δ-related to $\{1\}$ is X—which is not the case.

Nonetheless, (Pgrel) has some consequences for the other axioms. In order to give a more detailed view, (Pgrel) is divided into two sub-axioms.

(Pgreln) For all $a, b \subseteq X$: If $\delta(a,b)$, then $a \cap b^{\Uparrow\delta} \neq \emptyset$.

The added "n" stands for "necessary condition" as a necessary condition is specified for δ.

(Pgrels) For all $a, b \subseteq X$: $\delta(a,b)$ if $a \cap b^{\Uparrow\delta} \neq \emptyset$.

The "s" stands for sufficient condition.

Proposition 4. *The following entailment relations hold:*
1.) (Pgreln), (Pgrels) $\vDash$ *(P3) and 2.) (Pgrels)* $\vDash$ *(P4)*

So, with (Pgreln) and (Pgrels) the axiom (P3) becomes redundant, and (Pgrels) already entails (P4). (Pgrels) is already entailed by (P4). And hence (Pgrels) and (P4) are equivalent.

Proposition 5. *(P4)* $\vDash$ *(Pgrels)*

A simple consequence of axioms (Pgreln), (P2), (P4) is the monotonicity of the upshift operator.

Proposition 6. *If the axioms (Pgreln), (P2), (P4) hold, then monotonicity holds: For all* $a \subseteq b$ *one has* $a^{\Uparrow\delta} \subseteq b^{\Uparrow\delta}$.

So, the upshift operator $\cdot^{\Uparrow\delta}$ fulfills one of the conditions of a closure operator in a topological sense. But $\cdot^{\Uparrow\delta}$ is not a closure operator, not even a pre-closure/Cech-operator, i.e., it does not fulfill the following conditions for an operator $f : \mathsf{Pot}(X) \longrightarrow \mathsf{Pot}(X)$: (i) $f(\emptyset) = \emptyset$; (ii) $a \subseteq f(a)$; (iii) $f(a \cup b) = f(a) \cup f(b)$. (Here, $\mathsf{Pot}(X)$ = the power set of X). Condition (i) is not fulfilled as it is not defined for empty sets—but this could be remedied. Condition (ii) is fulfilled (under (P4)), but Condition (iii) states distributivity w.r.t. the union of sets.

5.2 Alignment of Upshift Close-ups

The upshift operator is intended to produce cells only. One aspect of this property is captured by the following nestedness condition.

(Pnested) For $a, b \subseteq X$: Either $a^{\Uparrow\delta} \subseteq b^{\Uparrow\delta}$ or $b^{\Uparrow\delta} \subseteq a^{\Uparrow\delta}$ or $a^{\Uparrow\delta} \cap b^{\Uparrow\delta} = \emptyset$.

As mentioned above, $\cdot^{\Uparrow_\delta}$ is not a closure operator. Nonetheless, one can state the following axioms characterizing the behavior of the double application of the operator—replacing idempotence—and characterizing the outcome of applying it to a union of sets—replacing distributivity over unions of sets.

(Pdoubleshift) If $a^{\Uparrow_\delta} \subsetneq b^{\Uparrow_\delta}$, then $a^{\Uparrow_\delta{}^{\Uparrow_\delta}} \subseteq b^{\Uparrow_\delta}$.

The axiom (Pdoubleshift) states that if the upshift of a is properly contained in a cell (the upshift of b), then another upshift application will keep it in this cell.

(Punionshift) If $a^{\Uparrow_\delta{}^{\Uparrow_\delta}} = b^{\Uparrow_\delta{}^{\Uparrow_\delta}}$ and $a^{\Uparrow_\delta} \not\subseteq b^{\Uparrow_\delta}$ and $b^{\Uparrow_\delta} \not\subseteq a^{\Uparrow_\delta}$, then $(a \cup b)^{\Uparrow_\delta} = a^{\Uparrow_\delta{}^{\Uparrow_\delta}{}^{\Uparrow_\delta}} = b^{\Uparrow_\delta{}^{\Uparrow_\delta}{}^{\Uparrow_\delta}}$.

Proposition 7. *All* sr_{pc} *over a normal partition chain pc fulfill (Pdoubleshift) and (Punionshift).*

The axioms above do not capture the effect of the $\tilde{\cdot}$ operator, which makes spatially relatedness being determined by its underlying cells. The main observation here is given by the following axiom. Intuitively, it says that subsets of two sets which are not upshift comparable lead to the same upshift.

(Pcelldet) If $a^{\Uparrow_\delta} \not\subseteq b^{\Uparrow_\delta}$ and $b^{\Uparrow_\delta} \not\subseteq a^{\Uparrow_\delta}$, then for all $a' \subseteq a$ and $b' \subseteq b$ (with $a', b' \neq \emptyset$) it follows that $(a' \cup b')^{\Uparrow_\delta} = ({}^{\Uparrow_\delta} a \cup b)$.

Proposition 8. *All* sr_{pc} *over a normal partition chain pc fulfill (Pcelldet).*

5.3 Isolated Points

An interesting point regarding $\cdot^{\Uparrow_\delta}$ is that it may contain fixed points or *isolated points*—as they are denoted in the following. In fact, for normal partition chains in which you may have sets a that occur on more than one level, lets call them *pc-fixpoints*, it holds that $a^{\Uparrow_{\mathsf{sr}_{pc}}} = a$: $a = a^{\Uparrow, pc} \stackrel{(Prop.3)}{=} a^{\Uparrow_{\mathsf{sr}_{pc}}}$.

Definition 5 (upshift-isolated). *A set* $a \subseteq X$ *is upshift isolated,* $\mathsf{uiso}(a)$*, iff* $a^{\Uparrow_\delta} = a$*.*

Now let us look again at points a in a normal partition pc that are pc-fixpoints. Another property these sets have is the following: If $\mathsf{sr}_{pc}(x, a)$, then $a \cap x \neq \emptyset$. Hence one may define the following equivalent notion of isolation:

Definition 6 (set-isolated). *A set* $a \subseteq X$ *is set-isolated,* $\mathsf{siso}(a)$*, iff: For all* $x \subseteq X$ *: if* $\delta(x, a)$*, then* $x \cap a \neq \emptyset$*.*

A simple observation is that these notions are the same if (Pgreln) and (P4) are fulfilled.

Proposition 9. $(Pgreln), (P4) \models \forall a.\mathsf{uiso}(a) \leftrightarrow \mathsf{siso}(a)$.

5.4 Splittings

In general, sr_{pc} relations do not fulfill the other direction in axiom (P2*) which states that if a is δ-related to $b \cup c$, then a is δ-related to b or c. Following [11], call the pair (b, c) with $b \cap c = \emptyset$ an *irregular split* of $b \cup c$ w.r.t. a. The main observation is that for any a there can be at most one irregular split.

Proposition 10. *For* sr_{pc}*, every set* a *has at most one irregular split.*

This property will now be formulated as an axiom over δ:

(PirrSplit) For δ, every a has at most one irregular split.

Any relation δ fulfilling (P2) and (PirrSplit) has a partition of X into cells which can serve as the cells of level 0. The crucial concept is the following.

Definition 7 (cell-equivalence). *For all* $x, y \in X$ *let the relation of cell-equivalence,* $\sim_0$ *for short, be defined by*

$$x \sim_0 y \textit{ iff } \{x\} \,{}^{\bullet}\!\!\sim \{x, y\} \textit{ and } \{y\} \,{}^{\bullet}\!\!\sim \{x, y\} \tag{4}$$

The cell-equivalence relation is indeed an equivalence relation:

Proposition 11. *Assume* δ *fulfills (P2) and (PirrSplit). Then the relation* $\sim_0$ *is an equivalence relation, i.e., it is symmetric, transitive, and reflexive.*

Actually, using the same proof idea, it is possible to prove the following theorem, which generalizes the result of the proposition.

Theorem 1. *For all subsets* $b_1, b_2 \subseteq [x]_{\sim_0}$*:* $b_1 \,{}^{\bullet}\!\!\sim b_2$*.*

So, this result gives the base on which to build the partition chain, namely the partition consisting of cells $[x]_{\sim_0}$.

6 Representation Theorem

This section gives the proof for the representation theorem for those spatial relations that are based on strict partition chains. A problem on building further cells upon cells $[x]_{\sim_0}$ are isolated sets. Hence, we explicitly exclude isolated sets.

(Pnoiso) For every $a \subsetneq X$ one has: $a \neq a^{\Uparrow\delta}$.

The main problem in representing spatially relatedness is to capture the fact that all paths from the root to the leaves in the pc have the same length. So consider the following notion of rank for any binary relation δ on X.

Definition 8 (rank). *For any* $a \in \mathsf{Pot}(X) \setminus \{\emptyset\}$ *we define by induction on* $n \in \mathbb{N}$*:* $a^0 = a$ *and* $a^{n+1} = a^{n\Uparrow\delta}$*. Then the* rank *of* a *is:*

$$r(a) = \begin{cases} m & \textit{s.t. there is } m' \textit{ with } a^{m'} = X \textit{ and} \\ & m \textit{ is the minimal one from the } m' \\ \infty & \textit{else} \end{cases}$$

The second case comes into play when there are isolated sets. Now one can formulate the following axiom which says that every pair of singleton sets $\{x\}$, $\{y\}$ over the domain X have the same rank

(Psamerank) For all $x, y \in X$: $r(\{x\}) = r(\{y\})$.

Proposition 12. sr_{pc} *over a strict partition chain pc fulfills (Psamerank).*

With these additional axioms, the representation theorem for spatial relations generated by strict partition chains can be proved.

Theorem 2. *If δ fulfills (P1), (P2), (Pgreln), (Pgrels), (Pnoiso), (PirrSplit), (Psamerank), (Pcelldet), (Pdoubleshift), and (Punionshift), then there is a strict pc, such that $\delta = \mathsf{sr}_{pc}$.*

Together with the propositions proved before one gets the following corollary.

Corollary 1. *A binary relation δ fulfills (P1), (P2), (Pgreln), (Pgrels), (Pnoiso), (PirrSplit), (Psamerank), (Pcelldet), (Pdoubleshift), and (Punionshift) if and only if there is a strict pc, such that $\delta = \mathsf{sr}_{pc}$.*

7 Related Work

The idea of a scaling context for spatial relations (more concretely: nearness relations) goes back to the work of [16]. This work and following work [3,8,17] do not deal with axiomatic characterizations as done in this paper.

The definition of spatially relatedness in this paper follows a general "information processing" strategy that can be found in different areas of computer science. Belief revision [1] is concerned with the general task of integrating a new piece of information a into a knowledge base b. If a is not compatible (associate: spatially related) to b, then one weakens b to a set b' with $b \models b'$ and $b' \cup \{a\} \not\models \bot$ by throwing out elements from b. The KB b' is less strict and thus less informative than b (associate: b' is becoming more similar to X). A similar situation appears in the area of abduction, where one has to find explanations for observations [7]. In most cases, observations cannot be deduced from the theory or facts at hand, but have to be found within a space of possible explanations. The idea is to keep the creativity effort needed as low as possible—going only a minimal step upwards in the explanation space.

The underlying structure of sr_{pc} are partition chains, which are special trees. The work of [14] and [6] focuses on the dynamics of such tree structures, called adjacency trees. In contrast, this paper uses these tree structures as a basis for a spatially relatedness definition and gives a full axiomatic characterization.

8 Conclusion and Outlook

The presented work gave a fine-grained semantical analysis of spatially relatedness based on partition chains—resulting in a representation theorem for the special case where the partition chain is strict. So, in fact, the work presented in this paper completed the characterization of the spatially relatedness relations in [10,11]. Users of information systems such as agents that move, act and plan in an environment according to some internal qualitative spatial map or semantic web systems [2] relying on spatio-temporal ontologies may benefit from this representation result because it completely characterizes the spatial relations at hand.

More concretely, if an agent or a query answering system would rely only on axiomatic characterizations of a spatially relatedness relation, then it would have to incorporate a deduction engine in order to do planning or query answering: Because only by considering all entailments of the axioms for spatially relatedness relations, the agent will guarantee that he will reach all possible plan configurations or all possible answers, respectively. On the other hand, knowing that the axioms for spatially relatedness have exactly one model (modulo renaming of the domain elements), the agent/the system may directly work with the model and apply, e.g., model checking—which is usually more feasible regarding complexity than calculating the deductive closure of a set of axioms.

Another benefit of the representation theorem for an AI planning or query answering agent is the possibility to focus on nested partition relations. In many situations these kinds of structures are not dynamic but sometimes may change due to some natural operations such as: two regions merge to an upper region, or an additional level of regions is established etc. (see [11], where these kinds of change are called global change). The change of the underlying partition structure leads to changes of the induced spatial relation. But one still knows that also the new spatial relation induced by the new nested partition chain still fulfills all properties as described by the axioms.

Regarding future work we note first that within the representation theorem a construction for strict partition chains on top of the upshift operator was used—relying on the exclusion of isolated sets. Spatial relations based on normal partition chains may have isolated sets, hence a different construction is called for. Moreover, this paper presumed finite domains X. This causes no problem as long as the upper structure of partitions is finite.

Further future work concerns the adaptations to overlapping hierarchies of regions. This is needed to model spatially relatedness induced by micro-functional areas with natural borderlines. Such areas (such as a forest, a valley, etc.) clearly have an influence on spatially relatedness but are not necessarily mutually disjoint.

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A Appendix: Proofs

Proof of Proposition 3

Clearly we have $(b)^{\Uparrow, pc} \subseteq (b)^{\Uparrow_{\mathsf{sr}_{pc}}}$: $(b)^{\Uparrow, pc}$ contains all cells b' on the same level as b for which $b \overset{\bullet}{\sim} b'$, hence $b' \subseteq (b)^{\Uparrow_{\mathsf{sr}_{pc}}}$. For the other direction assume for contradiction that there is some a with $b \overset{\bullet}{\sim} a$ and $a \nsubseteq (b)^{\Uparrow, pc}$. Let $a' := a \setminus (b)^{\Uparrow, pc}$. It holds that $\mathsf{sr}(a', a)$, as $a' \cap a \neq \emptyset$. Because of $b \overset{\bullet}{\sim} a$ it follows that also $\mathsf{sr}(a', b)$ contradicting $a' \cap (b)^{\Uparrow, pc} = \emptyset$.

Proof of Proposition 4

1. Let $\delta(a, c)$ or $\delta(b, c)$. Because of (Pgreln) it follows that $a \cap c^{\Uparrow_\delta} \neq \emptyset$ or $b \cap c^{\Uparrow_\delta} \neq \emptyset$. Hence $(a \cup b) \cap c^{\Uparrow_\delta} \neq \emptyset$ and with (Pgrels) it follows that $\delta(a \cup b, c)$. The other direction is proved similarly: Assume $\delta(a \cup b, c)$, then because of (Pgreln) it follows $(a \cup b) \cap c^{\Uparrow_\delta} \neq \emptyset$, i.e., $a \cap c^{\Uparrow_\delta} \neq \emptyset$ or $b \cap c^{\Uparrow_\delta} \neq \emptyset$ which with (Pgrels) entails $\delta(a, c)$ or $\delta(b, c)$.
2. Let $a \cap b \neq \emptyset$. In particular $b \neq \emptyset$. We have $b \overset{\bullet}{\sim} b$, hence $b \subseteq b^{\Uparrow_\delta}$. Hence $a \cap b^{\Uparrow_\delta} \neq \emptyset$, so with (Pgrels) it follows that $\delta(a, b)$. The same argument works for the roles of a and b exchanged.

Proof of Proposition 5

Let $a \cap b^{\Uparrow_\delta} \neq \emptyset$. Then there is a c such that $c \overset{\bullet}{\sim} b$ and $a \cap c \neq \emptyset$. Because of (P4) we get $\delta(a, c)$. So due to the definition of $\overset{\bullet}{\sim}$ we must also have $\delta(a, b)$.

Proof of Proposition 6

Let $a \subseteq b$. Let $x \in a^{\Uparrow_\delta}$. So there is a c with $c \overset{\bullet}{\sim} a$ and $x \in c$. We show $c \subseteq b^{\Uparrow_\delta}$ (and so also $x \in b^{\Uparrow_\delta}$). Assume not, then there is a subset of $c' \subset c$ such that $c' \cap b^{\Uparrow_\delta} = \emptyset$. Because of (Pgreln) it follows that not $\delta(c', b)$. Because of (P2) also not $\delta(c', a)$, and because of $c \overset{\bullet}{\sim} a$ also not $\delta(c', c)$, but this contradicts (P4).

Proof of Proposition 7

Due to Proposition 3 we can assume that $\cdot^{\Uparrow_\delta} = (\cdot)^{\Uparrow^{\mathsf{sr}_{pc}}} = (\cdot)^{\Uparrow, pc}$. Now, let $a^{\Uparrow_\delta} \subsetneq b^{\Uparrow_\delta}$, so the pc-cell $a^{\Uparrow_\delta}$ is a proper subset of the cell $b^{\Uparrow_\delta}$. But then an additional application of $\cdot^{\Uparrow_\delta} = (\cdot)^{\Uparrow, pc}$ amounts to a shift in *pc* which must be a cell contained in or the same as the cell $b^{\Uparrow_\delta}$. This shows (Pdoubleshift).

In order to prove (Punionshift), let $a^{\Uparrow_\delta \Uparrow_\delta} = b^{\Uparrow_\delta \Uparrow_\delta} \neq X$ and $a^{\Uparrow_\delta} \nsubseteq b^{\Uparrow_\delta}$ and $b^{\Uparrow_\delta} \nsubseteq a^{\Uparrow_\delta}$. So the pc-cells $a^{\Uparrow_\delta}$ and $b^{\Uparrow_\delta}$ are not on the same path from the leaves to the root X in pc, but their upper shifts are. So, that means that $\widetilde{a \cup b}$ must be a cell that contains $a^{\Uparrow_\delta}$ and $b^{\Uparrow_\delta}$. As it is the smallest such cell we get $\widetilde{a \cup b} = a^{\Uparrow_\delta \Uparrow_\delta} = b^{\Uparrow_\delta \Uparrow_\delta}$. Hence $a \cup b^{\Uparrow_\delta} = a^{\Uparrow_\delta \Uparrow_\delta \Uparrow_\delta} = b^{\Uparrow_\delta \Uparrow_\delta \Uparrow_\delta}$.

Proof of Proposition 8

Let $a^{\Uparrow\delta} \not\subseteq b^{\Uparrow\delta}$ and $b^{\Uparrow\delta} \not\subseteq a^{\Uparrow\delta}$. So the pc-cells $a^{\Uparrow\delta}$ and $b^{\Uparrow\delta}$ are not on the same path from the leaves to the root X in pc. That means that $\widetilde{a \cup b}$ must be a cell whose underlying set contains $a^{\Uparrow\delta}$ and $b^{\Uparrow\delta}$. Now consider $\widetilde{(a' \cup b')}$. We prove $us((\widetilde{a' \cup b'})^{\uparrow,pc}) = us((\widetilde{a \cup b})^{\uparrow,pc})$. As $a' \cup b' \subseteq a \cup b$, it follows that $us(\widetilde{a' \cup b'}) \subseteq us(\widetilde{a \cup b})$, so we know that the cell $\widetilde{a' \cup b'}$ must be under the cell $\widetilde{a \cup b}$. Now assume that $\widetilde{a \cup b}$ is strictly above the cell $\widetilde{a' \cup b'}$. As $a^{\Uparrow\delta}$ and $b^{\Uparrow\delta}$ are incomparable it must be the case that $us(\widetilde{a' \cup b'}) \supseteq a^{\Uparrow\delta} \cup b^{\Uparrow\delta}$. Hence $us(\widetilde{a' \cup b'}) \supseteq us(\widetilde{a^{\Uparrow\delta} \cup b^{\Uparrow\delta}}) \supseteq us(\widetilde{a \cup b})$ which means that the underlying set of $\widetilde{a \cup b}$ is contained in the underlying set of the cell $\widetilde{a' \cup b'}$. With the assertion proven before this would mean that $\widetilde{a \cup b}$ and $\widetilde{a' \cup b'}$ have the same underlying sets. Could it be the case that the level of $\widetilde{a \cup b}$ is strictly higher than that of $\widetilde{a' \cup b'}$? No, because $\widetilde{a \cup b}$ the smallest low level cell containing $a \cup b$ and this must be the level of $\widetilde{a' \cup b'}$ as the underlying set is the same as that of $\widetilde{a \cup b}$.

Proof of Proposition 9

Assume $\mathsf{uiso}(a)$. In order to show $\mathsf{siso}(a)$, let $\delta(x, a)$, then (Pgreln) says that $x \cap a^{\Uparrow\delta} \neq \emptyset$, but $a^{\Uparrow\delta} = a$, so $x \cap a \neq \emptyset$. Now assume $\mathsf{siso}(a)$. We have to show, $a^{\Uparrow\delta} = a$, in effect only $a^{\Uparrow\delta} \subseteq a$. So let $b \sim^{\bullet} a$, that means that all b have the same set of incoming δ edges as a. We have to show $b \subseteq a$. Assume otherwise, let $e = b \setminus a$. We have $\delta(e, b)$. So we must also have $\delta(e, a)$. But as siso, this entails $e \cap a \neq$, contradiction.

Proof of Proposition 10

Assume $\mathsf{sr}(a, b \uplus c)$ and not $\mathsf{sr}(a, b)$ and not $\mathsf{sr}(a, c)$. As $us(b^{\Uparrow}) \cap a = \emptyset$ and $us((c^{\Uparrow}) \cap a = \emptyset$ but $us((b \uplus c)^{\Uparrow}) \cap a \neq \emptyset$, we have $us(b^{\Uparrow}) \cup us(c^{\Uparrow}) \subsetneq us((b \uplus c)^{\Uparrow})$. Hence it follows that $us((b^{\Uparrow}) \neq us(c^{\Uparrow})$, because otherwise one would have $us(b^{\Uparrow}) \cup us(c^{\Uparrow}) = us((b \uplus c)^{\Uparrow})$. Now, let $b \uplus c = b' \uplus c'$ where $b' \neq b$ and $c \neq c'$. One of b', c' must have elements of both b and c. W.l.o.g let us assume it is b'. That means that $\tilde{b'} = \widetilde{b \cup c}$ and hence $\mathsf{sr}(a, b')$.

Proof of Proposition 11

Clearly reflexivity and symmetry hold for $\sim_0$. We have to show transitivity. So let $x \sim_0 y$ and $y \sim_0 z$. We have to show that $x \sim_0 z$. For contradiction assume $x \not\sim_0 z$. Then either not $\{x\} \,{}^{\bullet}\!\!\sim \{x, z\}$ or not $\{z\} \,{}^{\bullet}\!\!\sim \{x, z\}$. Assume it is not $\{x\} \,{}^{\bullet}\!\!\sim \{x, z\}$. (The other case is handled symmetrically.) As (P2) holds, this can be the case only if there is a set a such that $\delta(a, \{x, z\})$ but not $\delta(a, \{x\})$. The latter together with the assumptions that $x \sim_0 y$ and $y \sim_0 z$ implies, that not $\delta(a, \{x, y\})$ and not $\delta(a, \{y\})$ and not $\delta(a, \{y, z\})$ and not $\delta(a, \{z\})$. But now, as $\delta(a, \{x, z\})$, axiom $(P2)$ implies also $\delta(a, \{x, z, y\})$. But this means that we have two different irregular splits of $\{x, z, y\}$ w.r.t. a, namely $\{x\} \uplus \{z, y\}$ and $\{z\} \uplus \{x, y\}$. This contradicts Axiom (PirrSplit), hence we may conclude that $\sim_0$ is indeed an equivalence relation and that for all $x \in X$, the equivalence class $[x]_{\sim_0}$ is defined.

Proof of Theorem 1

Let $b_1, b_2 \subseteq [x]_{\sim_0}$. If b_1, b_2 are singletons, then the assertion follows directly from the assumption. Now assume that one of them contains two elements, say it is $b_1 = \{e, f\}$ and $b_2 = \{d\}$. One knows

$$b_1 = \{e, f\} \;{}^{\bullet}\!\!\sim \{f\} \;{}^{\bullet}\!\!\sim \{f, d\} \;{}^{\bullet}\!\!\sim \{d\} = b_2$$

If $b_2 = \{d_1, d_2\}$ then we have

$$b_1 = \{e, f\} \;{}^{\bullet}\!\!\sim \{f\} \;{}^{\bullet}\!\!\sim \{f, d_1\} \;{}^{\bullet}\!\!\sim \{d_1\} \;{}^{\bullet}\!\!\sim \{d_1, d_2\} = b_2$$

Now a more restricted version of the proposition is proved: Namely for all $b \subseteq [x]_{\sim_0}$ we have for all $x \in b$: $x \;{}^{\bullet}\!\!\sim b$. This is true for b of size 1 and 2. So assume for induction that it holds for all b of size n and assume that b has size $n+1$, e.g., $b = b' \cup \{e\}$. Take an arbitrary $z \in b' \cup \{e\}$. In the first case say $z \in b'$. we have $z \;{}^{\bullet}\!\!\sim b'$ by induction and $z \;{}^{\bullet}\!\!\sim e$.If we had not $z \;{}^{\bullet}\!\!\sim b' \cup \{e\}$ this could only be the case because there exist f with $\delta(f, b' \cup \{e\})$ but not $\delta(f, z)$, that would also mean that not $\delta(f, b')$ and not $\delta(f, e)$. So we get an irregular split of $b' \cup \{e\}$ for f. As b' contains at least two elements, say $g \in b'$ and $g \neq e$, we consider now $b' \setminus \{g\} \cup \{e, g\}$. Now it must be not $\delta(f, b' \setminus \{g\})$ and not $\delta(f, \{e, g\})$ as $e \;{}^{\bullet}\!\!\sim \{e, g\}$. So we end up with more than one irregular splitting, contradiction. Now of course, take b_1 and b_2 arbitrarily. Then $b_1 \;{}^{\bullet}\!\!\sim x$ for any $x \in b_1$ and $b_2 \;{}^{\bullet}\!\!\sim y$ for any $y \in b_2$, but $x \;{}^{\bullet}\!\!\sim y$, hence $b_1 \;{}^{\bullet}\!\!\sim b_2$.

Proof of Theorem 2

Cells of level 0 are constructed as $[x]_{\sim_0}$ which is possible due to Prop. 11. On top of these one constructs per recursion other partitions, showing that these indeed are partitions and that all cells built have the same rank.

Assume that one has already constructed cells up to level n, i.e., one has a partition of sets on level n and a corresponding equivalence relation $\sim_n$. In case the n^{th} partition consists only of the set X, the construction is finished. Otherwise one defines the cells on level $n+1$ as sets of the forms $a^{\Uparrow_\delta}$, where a is a cell on level n. We have to show that theses indeed make up a set partition. Clearly, these sets cover the whole set X: Because there are no isolated points different from X due to (Pnoiso), we have $a \subsetneq a^{\Uparrow_\delta}$. Due to (Pnested) we know that the sets on level n+1 are going to be aligned. But this does not exclude that one set $a^{\Uparrow_\delta}$ is a proper subset of another set $b^{\Uparrow_\delta}$ for sets a, b on level n, i.e., assume for contradiction that $a^{\Uparrow_\delta} \subsetneq b^{\Uparrow_\delta}$. Due to (Pdoubleshift) it follows that $a^{\Uparrow_\delta \Uparrow_\delta} \subseteq b^{\Uparrow_\delta}$. Now take any $x \in a$ and $y \in b$. We consider two cases: n = 0. So a, b are cells on level 0. Because of Thm. 1 we know that $x^{\Uparrow_\delta} = a^{\Uparrow_\delta}$ and $y^{\Uparrow_\delta} = b^{\Uparrow_\delta}$. But then we get the following (in)equalities:

$$\begin{aligned} r(\{x\}) &\overset{\text{(Prop. 1)}}{=} r(a) \overset{\text{(Def. of } r(\cdot))}{=} 1 + r(a^{\Uparrow_\delta}) \overset{\text{(Def. of } r(\cdot))}{=} 2 + r(a^{\Uparrow_\delta \Uparrow_\delta}) \\ &\overset{\text{(Pdoubleshift)}}{\geq} 2 + r(b^{\Uparrow_\delta}) > 1 + r(b^{\Uparrow_\delta}) \overset{\text{(Def. of } r(\cdot))}{=} r(b) \overset{\text{(Def. of } r(\cdot))}{=} r(y^{\Uparrow_\delta}) \end{aligned}$$

So, one would get $r(\{x\}) > r(\{y\})$ contradicting (Phomrank). So, it holds that all cells of level 0 have the same rank.

If $n > 0$, then a and b have the forms $a = a'^{\Uparrow_\delta}$ and $b = b'^{\Uparrow_\delta}$ for cells a', b' on level $n - 1$. We may assume that these have the same rank (per induction.) Similar as for case $n = 0$ one calculates the inequality $r(a') > r(b')$, which contradicts the induction hypothesis.

Now assume pc is the partition chain resulting from this construction. We have to show that $\delta = \mathsf{sr}_{pc}$. One has to show for all a, b that $\delta(a, b)$ iff $\mathsf{sr}_{pc}(a, b)$. Assume that b is a cell in pc. Then $us(\tilde{b}) = b$ in pc, and $(b)^{\Uparrow, pc} = (\tilde{b})^{\uparrow, pc} = us(\tilde{b})^{\Uparrow_\delta} = b^{\Uparrow_\delta}$. So with (Pgreln) and (Pgrels) one gets $\delta(a, b)$ iff $\mathsf{sr}_{pc}(a, b)$.

Now assume that b is an arbitrary set. Let $\tilde{b}$ be the cell in pc containing b. If one can show that $b^{\Uparrow_\delta} = us(\tilde{b})^{\Uparrow_\delta}$, then one can use the same argument as above for the case $b = us(\tilde{b})$. Now show by induction on the level N of $\tilde{b}$ that $b = us(\tilde{b})$.

Case $N = 0$: Here $\tilde{b}$ is a cell on level zero. Because of Prop. 1 we know that $b \overset{\bullet}{\sim} us(\tilde{b})$, hence $b^{\Uparrow_\delta} = us(\tilde{b})^{\Uparrow_\delta}$.

Case $N > 0$: Two sub-cases are distinguished, $N = 1$ and $N > 1$. Assume first that $N = 1$. So let $us(\tilde{b})$ have level 1. Then b is covered by a set $\{c_1, \ldots, c_k\}$ of cells of level 0. We argue that it cannot be the case that $b^{\Uparrow_\delta} \subseteq us(\tilde{b})$: take $x_1 \in c_1, x_2 \in c_2$. As x_1, x_2 are in different cells c_1, c_2 of level 0 we know that not ($x_1 \overset{\bullet}{\sim} \{x_1, x_2\}$ and $x_2 \overset{\bullet}{\sim} \{x_1, x_2\}$). Assume without loss of generality that not $x_1 \overset{\bullet}{\sim} \{x_1, x_2\}$. That means that there is a z such that $\delta(z, \{x_1, x_2\})$ but not $\delta(z, \{x_1\})$. But one has $b^{\Uparrow_\delta} \supseteq \{x_1, x_2\}^{\Uparrow_\delta} \supseteq x_1{}^{\Uparrow_\delta} = c_1{}^{\Uparrow_\delta} = us(\tilde{b})$. Now it cannot be the case that $b^{\Uparrow_\delta} = us(\tilde{b})$, because then $\{x_1, x_2\}^{\Uparrow_\delta} = us(\tilde{b}) = x_1{}^{\Uparrow_\delta}$. As $\delta(z, \{x_1, x_2\})$, this means by (Pgreln) that $z \in \{x_1, x_2\}^{\Uparrow_\delta} = x_1{}^{\Uparrow_\delta}$. By (Pgrels) it then follows $\delta(z, x_1)$—contradiction. Hence, $b^{\Uparrow_\delta} \subsetneq us(\tilde{b})$. But still we could have $b^{\Uparrow_\delta} \subsetneq us(\tilde{b})^{\Uparrow_\delta}$. But then due to nestedness (Pnested) $b^{\Uparrow_\delta}$ must contain properly at least one cell of level 0, say d. So one has $d^{\Uparrow_\delta} \subsetneq b^{\Uparrow_\delta}$, hence with (Pdoubleshift) it follows that $d^{\Uparrow_\delta \Uparrow_\delta} \subseteq b^{\Uparrow_\delta} \subsetneq us\tilde{b}^{\Uparrow_\delta}$. But as d is a cell of level 1, we must have $d^{\Uparrow_\delta} = us(\tilde{b})$ leading to a contradiction. Hence $b^{\Uparrow_\delta} = \tilde{b}^{\Uparrow_\delta}$ follows.

Case $N > 1$. Let $\tilde{b}$ be a cell of level $N > 1$ and let b be covered by a set $\{c_1, \ldots, c_k\}$ of cells of level $N - 1 > 0$. The cells c_1, c_2 can be represented as $c_1 = c_1'^{\Uparrow_\delta}$ and $c_2 = c_2'^{\Uparrow_\delta}$ for $N-1$ level cells c_1', c_2'. We can choose c_1' and c_2' such that $b \cap c_i' \neq \emptyset$, i.e., there is $x_1 \in c_1' \cap B$ and $x_2 \in c_2' \cap B$. We have $c_1'^{\Uparrow_\delta} \not\subseteq c_2'^{\Uparrow_\delta}$ and $c_2'^{\Uparrow_\delta} \not\subseteq c_1'^{\Uparrow_\delta}$ and $c_1'^{\Uparrow_\delta \Uparrow_\delta} = c_2'^{\Uparrow_\delta \Uparrow_\delta} \neq X$. Using axiom (Punionshift) it follows that $c_1' \cup c_2'^{\Uparrow_\delta} = c_1'^{\Uparrow_\delta \Uparrow_\delta \Uparrow_\delta} = c_2'^{\Uparrow_\delta \Uparrow_\delta \Uparrow_\delta} = \tilde{b}^{\Uparrow_\delta}$. Because of (Pcelldet) it follows that $x_1 \cup x_2{}^{\Uparrow_\delta} = \tilde{b}^{\Uparrow_\delta}$ and so also $b^{\Uparrow_\delta} = \tilde{b}^{\Uparrow_\delta}$.