Intelligent Agents:
Web-mining Agents

Probabilistic Graphical Models

Propositional Modelling

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Probabilistic Graphical Models (PGMs)

1. Recap: **Propositional** modelling
   - Factor model, Bayesian network, Markov network
   - Semantics, inference tasks + algorithms + complexity
2. **Probabilistic relational models** (PRMs)
   - Parameterised models, Markov logic networks
   - Semantics, inference tasks
3. **Lifted inference**
   - LVE, LJT, FOKC
   - Theoretical analysis
4. **Lifted learning**
   - Recap: propositional learning
   - From ground to lifted models
   - Direct lifted learning
5. **Approximate Inference: Sampling**
   - Importance sampling
   - MCMC methods
6. **Sequential models & inference**
   - Dynamic PRMs
   - Semantics, inference tasks + algorithms + complexity, learning
7. **Decision making**
   - (Dynamic) Decision PRMs
   - Semantics, inference tasks + algorithms, learning
8. **Continuous Space**
   - Gaussian distributions and Bayesian networks
   - Probabilistic soft logic

Outline: 1. Recap: Propositional Modelling

A. Probabilistic modelling
- Full joint probability distribution
- Inference, complexity

B. Factorised modelling
- (Conditional) independences
- Factorisation

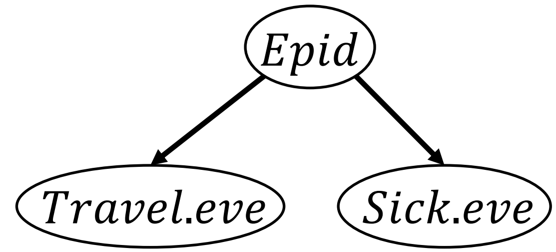
C. Inference algorithm
- Variable elimination (VE)
- Decomposition trees, complexity

Propositional Models

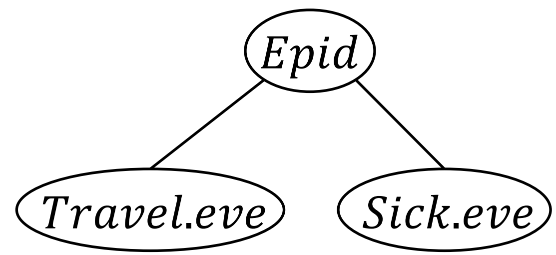
- Probabilistic description of a scenario under investigation

$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

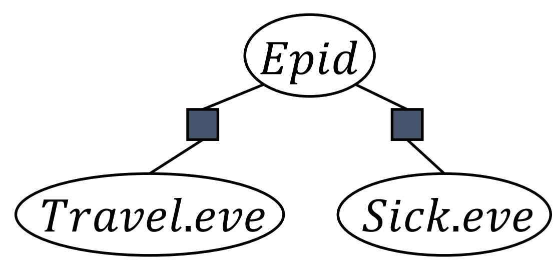
Full joint probability distribution



Bayesian network (BN)



Markov network (MN)

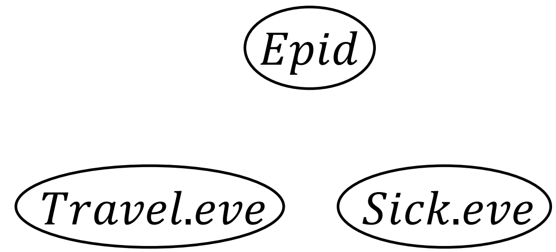


Factor graph (FG)

Random Variables

- Characterise scenario by set of random variables
  - $\boldsymbol{R} = \{R_1, \dots, R_n\}$
  - Often depicted as ellipses
  - E.g.,

$$\{Epid, Travel.eve, Sick.eve\}$$



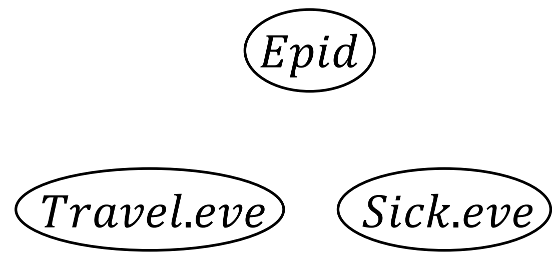
- Possible values a random variable can take = range
  - $\mathcal{R}(R) = \{v_1, \dots, v_m\}$
  - If $|\mathcal{R}(R)| = 2$, often called Boolean range
  - E.g.,

$$\mathcal{R}(Epid) = \mathcal{R}(Travel.eve) = \mathcal{R}(Sick.eve) = \{true, false\}$$

Events

- Observing or setting a random variable to a specific range value = event
  - $R = r, r \in \mathcal{R}(R)$
  - Shorthand:
    - If R clear from context, we write r instead of $R = r$
      - If $\mathcal{R}(R)$ Boolean, we write r for $R = true$ and $\neg r$ for $R = false$
  - E.g.,

$$Epid = true \qquad Epid = false$$
$$epid \qquad \neg epid$$



- Setting range values for a set of random variables, one value for each variable = compound event

Full Joint Probability Distribution

- 1 world = compound event for $\boldsymbol{R}$

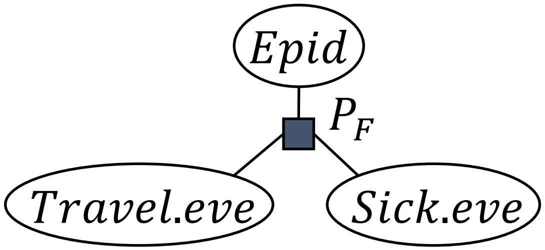
$$epid$$
$$\neg travel.eve$$
$$\neg sick.eve$$

- Specify a probability for a world

$$P(epid, \neg travel.eve, \neg sick.eve) = 0.05$$

How large is l?

- Joint probability distribution P_F over all (l) possible worlds
  - $\sum_{i=1}^{l} P(w_i) = 1$
    - w_i: compound event for $\boldsymbol{R}$



$Epid$	$Travel.eve$	$Sick.eve$	P
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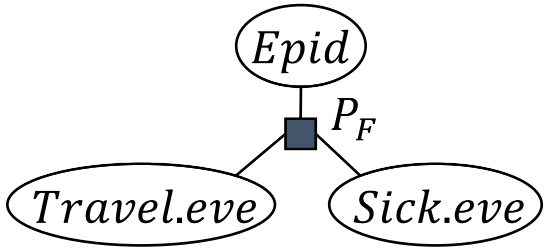
$true$	$true$	$true$	0.02

Space Complexity

- Joint probability distribution P_F over all (l) possible worlds
  - $\sum_{i=1}^{l} P(w_i) = 1$
    - w_i: compound event for $R \in \boldsymbol{R}$
- Space complexity: $O(r^n)$
  - $r = \max_{R \in \boldsymbol{R}} |\mathcal{R}(R)|$
  - $n = |\boldsymbol{R}|$
  - Exact size:

$$\prod_{R \in \boldsymbol{R}} |\mathcal{R}(R)|$$

Exponential in n!



$Epid$	$Travel.eve$	$Sick.eve$	P
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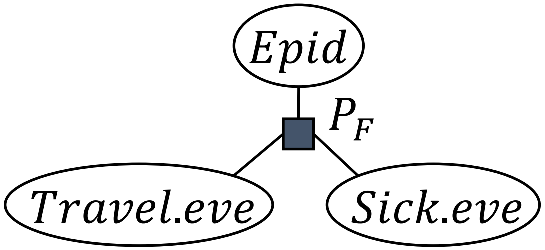
$true$	$true$	$true$	0.02

Inference Tasks

- Query Answering Problem
  - Compute an answer to a query given full joint probability distribution P_F
    - Query for a marginal (conditional) probability (distribution)
    - Query forms:
      - Marginal probability of events
      - Marginal probability distribution of random variables
      - Marginal conditional probability of events given events
      - Marginal conditional probability distribution of random variables given random variables or events
- Next slides
  - Syntax of queries
  - Solving an instance of a query answering problem
    - Preview: Eliminate all non-query terms

Marginal Queries

- Query for marginal probability (distribution) w.r.t. P_F
  - $P(\boldsymbol{S})$
    - $rv(\boldsymbol{S}) \subseteq \boldsymbol{R}$
      - $rv(.)$: shorthand notation to refer to random variables in an expression
  - $\boldsymbol{S}$: random variables (query for distribution) or events (query for probability)
  - E.g.,
    $P(epid)$
    $P(Epid, Travel.eve)$



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$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
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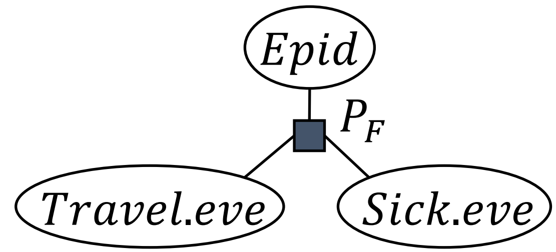
Answering Marginal Queries

- Given a query $P(\boldsymbol{S})$ w.r.t. P_F

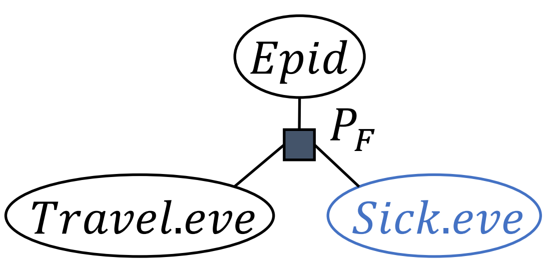


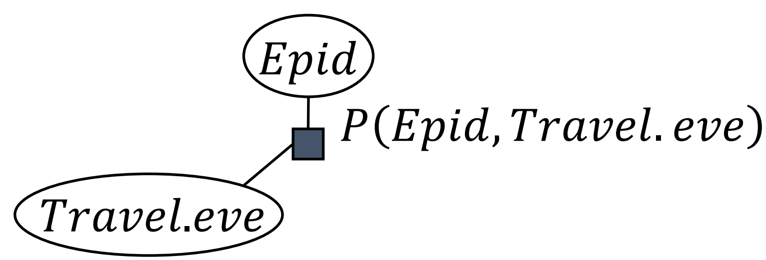
- Eliminate all non-query terms
  - $\boldsymbol{U} = \boldsymbol{R} \setminus rv(\boldsymbol{S})$
  - Given $R_1 \dots R_n \in \boldsymbol{U}$:

$$P(\boldsymbol{S}) = \sum_{v_1 \in \mathcal{R}(R_1)} \dots \sum_{v_n \in \mathcal{R}(R_n)} P_F(R_1 = v_1, \dots, R_n = v_n)$$

  - E.g.,

$$P(Epid, Travel.eve) \qquad \boldsymbol{U} = \{Sick.eve\}$$





Answering Marginal Queries

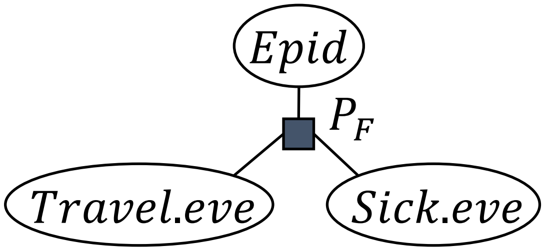
$$P(Epid, Travel.eve) = \sum_{v \in \mathcal{R}(Sick.eve)} P(Epid, Travel.eve, Sick.eve = v)$$

$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

$Epid$	$Travel.eve$	P
$false$	$false$	0.44
$false$	$true$	0.36
$true$	$false$	0.11
$true$	$true$	0.09

Answering Marginal Queries

- Given a query $P(\boldsymbol{S})$ w.r.t. P_F



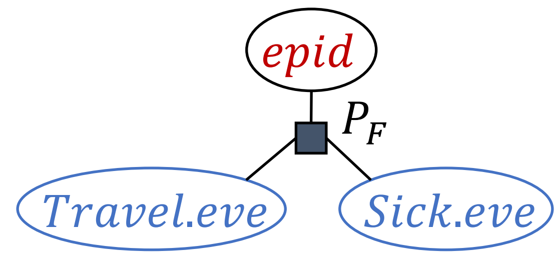
- Eliminate all non-query terms
  - $\boldsymbol{U} = \boldsymbol{R} \setminus rv(\boldsymbol{S})$
  - Given $R_1 \dots R_n \in \boldsymbol{U}$:

$$P(\boldsymbol{S}) = \sum_{v_1 \in \mathcal{R}(R_1)} \dots \sum_{v_n \in \mathcal{R}(R_n)} P_F(R_1 = v_1, \dots, R_n = v_n)$$

  - E.g.,
    $P(Epid, Travel.eve) \qquad \boldsymbol{U} = \{Sick.eve\}$
  - If $\boldsymbol{S}$ consists of events, consider only the cases where the events are true
  - E.g.,
    $P(epid) \qquad \boldsymbol{U} = \{Travel.eve, Sick.eve\}$

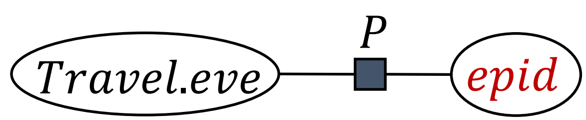
Answering Marginal Queries

$$P(epid) = \sum_{v_t \in \mathcal{R}(Travel.eve)} \sum_{v_s \in \mathcal{R}(Sick.eve)} P(epid, Travel.eve = v_t, Sick.eve = v_s)$$



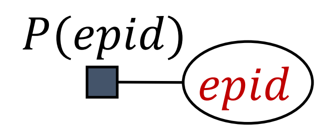
$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

+



$Epid$	$Travel.eve$	P
$true$	$false$	0.11
$true$	$true$	0.09

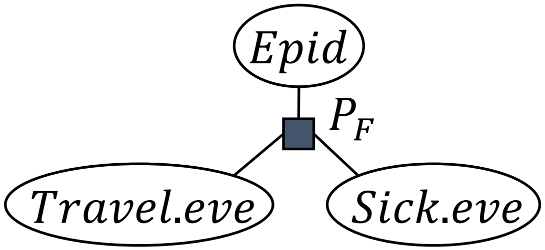
+



$Epid$	P
$true$	0.20

Conditional Queries

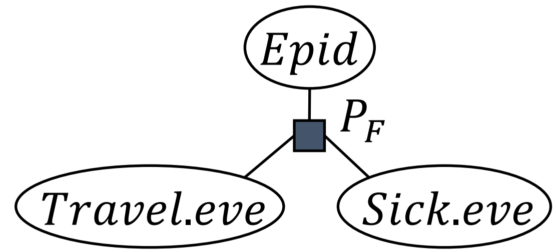
- Query for conditional marginal probability distribution w.r.t. P_F
  - $P(\boldsymbol{S}|\boldsymbol{T})$
    - $rv(\boldsymbol{S}, \boldsymbol{T}) \subseteq \boldsymbol{R}$
      - $rv(.)$: shorthand notation to refer to random variables in an expression
    - $\boldsymbol{S} \cap \boldsymbol{T} = \emptyset$
  - $\boldsymbol{S}$: as before
  - $\boldsymbol{T}$: random variables or events (considered observations, called evidence)



- E.g.,
  $P(Sick.eve|Epid)$
  $P(Epid|sick.eve)$

Answering Conditional Queries

- Given a query $P(\boldsymbol{S}|\boldsymbol{T})$ w.r.t. P_F
  - $P(\boldsymbol{S}|\boldsymbol{T}) = \frac{P(\boldsymbol{S},\boldsymbol{T})}{P(\boldsymbol{T})}$
  - $P(\boldsymbol{T})$ normalising constant



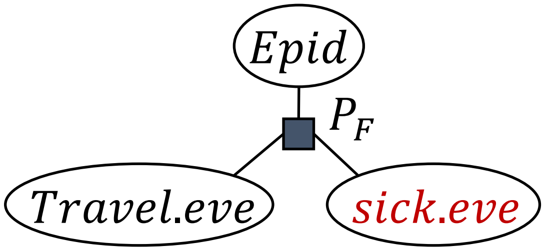
- Reduces to computing two marginal queries: $P(\boldsymbol{S},\boldsymbol{T})$ and $P(\boldsymbol{T})$
- Eliminate all non-query terms and normalise
  - $\boldsymbol{U} = \boldsymbol{R} \setminus rv(\boldsymbol{S},\boldsymbol{T})$
  - Given $R_1 \dots R_n \in \boldsymbol{U}$:

$$P(\boldsymbol{S}|\boldsymbol{T}) = \frac{1}{P(\boldsymbol{T})} \sum_{v_1 \in \mathcal{R}(R_1)} \dots \sum_{v_n \in \mathcal{R}(R_n)} P_F(R_1 = v_1, \dots, R_n = v_n, \boldsymbol{S}, \boldsymbol{T})$$

  - E.g., $P(Epid|sick.eve)$ $\boldsymbol{U} = \{Travel.eve\}$ $P(\boldsymbol{T}) = P(sick.eve)$

Answering Conditional Queries

- Given a query $P(\boldsymbol{S}|\boldsymbol{T})$ w.r.t. P_F
  - $P(\boldsymbol{S}|\boldsymbol{T}) = \frac{P(\boldsymbol{S},\boldsymbol{T})}{P(\boldsymbol{T})}$
  - $P(\boldsymbol{T})$ normalising constant



- Reduces to computing two marginal queries: $P(\boldsymbol{S},\boldsymbol{T})$ and $P(\boldsymbol{T})$
- Eliminate all non-query terms and normalise
  - $\boldsymbol{U} = \boldsymbol{R} \setminus rv(\boldsymbol{S},\boldsymbol{T})$
  - Given $R_1 \dots R_n \in \boldsymbol{U}$:

$$P(\boldsymbol{S}|\boldsymbol{T}) = \frac{1}{P(\boldsymbol{T})} \sum_{v_1 \in \mathcal{R}(R_1)} \dots \sum_{v_n \in \mathcal{R}(R_n)} P_F(R_1 = v_1, \dots, R_n = v_n, \boldsymbol{S}, \boldsymbol{T})$$

  - If $\boldsymbol{T}$ contains event $R = r$, set probabilities to 0 where $R \neq r$

$P(Epid|sick.eve)$

Called absorption

Answering Conditional Queries

$P(Epid|sick.eve)$

$\propto \sum_{\ldots ravel.eve)} P(Epid, Travel.eve = v, sick.eve)$

Proportional to

Dimension reduction!

$Epid$	$Travel.eve$	$Sick.eve$	P
~~$false$~~	~~$false$~~	~~$false$~~	~~0.20~~
$false$	$false$	$true$	0.24
~~$false$~~	~~$true$~~	~~$false$~~	~~0.28~~
$false$	$true$	$true$	0.08
~~$true$~~	~~$false$~~	~~$false$~~	~~0.05~~
$true$	$false$	$true$	0.06
~~$true$~~	~~$true$~~	~~$false$~~	~~0.07~~
$true$	$true$	$true$	0.02

P
~~0~~
0.24
~~0~~
0.08
~~0~~
0.06
~~0~~
0.02

$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$true$	0.24
$false$	$true$	$true$	0.08
$true$	$false$	$true$	0.06
$true$	$true$	$true$	0.02

$Epid$	$Travel.eve$	P
$false$	$false$	0.24
$false$	$true$	0.08
$true$	$false$	0.06
$true$	$true$	0.02

Answering Conditional Queries

$$P(Epid|\textcolor{red}{sick.eve}) \propto \sum_{v \in \mathcal{R}(Travel.eve)} P(Epid, Travel.eve = v, \textcolor{red}{sick.eve})$$

$Epid$	$Travel.eve$	P
$false$	$false$	0.24
$false$	$true$	0.08
$true$	$false$	0.06
$true$	$true$	0.02

+

$Epid$	P
$false$	0.32
$true$	0.08

normalise

$Epid$	P
$false$	$\frac{0.32}{0.32+0.08} = 0.8$
$true$	$\frac{0.08}{0.32+0.08} = 0.2$

Runtime Complexity

$$P(\boldsymbol{S}|\boldsymbol{T}) = \frac{1}{P(\boldsymbol{T})} \sum_{v_1 \in \mathcal{R}(R_1)} \dots \sum_{v_n \in \mathcal{R}(R_n)} P_F(R_1 = v_1, \dots, R_n = v_n, \boldsymbol{S}, \boldsymbol{T})$$

- Runtime complexity: $O(r^n)$
  - $r = \max_{R \in \boldsymbol{R}} |\mathcal{R}(R)|$
  - $n = |\boldsymbol{R}|$
  - Exact size:
    $$\prod_{R \in \boldsymbol{R}} |\mathcal{R}(R)|$$

= Space complexity

$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

Exponential Blowup!

$Man.war$	$Man.virus$	$Nat.flood$	$Nat.fire$	$Treat.eve.m_1$	$Treat.eve.m_2$	$Epid$	$Travel.eve$	$Sick.eve$	P
false	false	false	false	false	false	false	false	fals	
false	false	false	false	false	false	false	false		
false	false	false	false	false	false	false			
false	false	false	false	false	false	fal			
false	false	false	false	false	false				
false	false	false	false	false					
false	false	false	false	false					
false	false	false	false						
false	false	false	fals						
false	false	false							
false	false								
false	false								
false									
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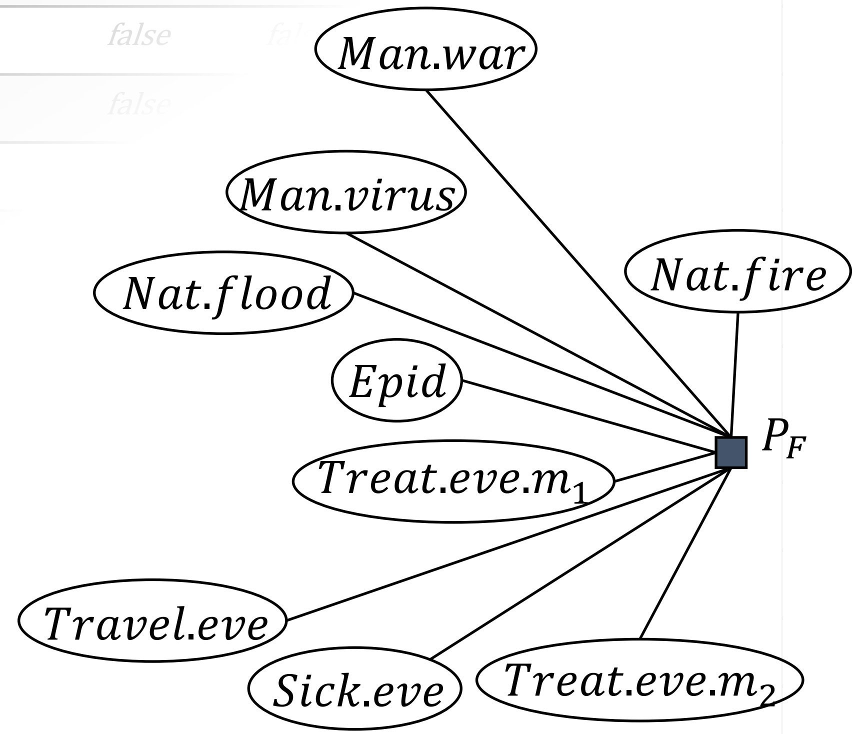


$2^9 = 512$ possible worlds

Interim Summary

- Full joint probability distribution
  - Over set of random variables
  - Space complexity
- Inference tasks
  - Query answering problem
  - Queries for marginal (conditional) probability (distribution)
  - Runtime complexity

Outline: 1. Recap: Propositional Modelling

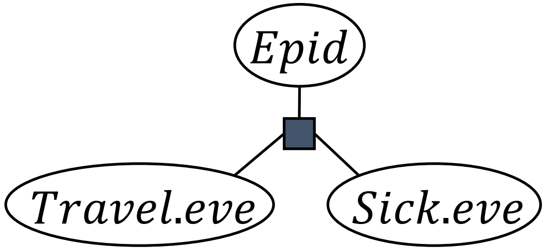
A. *Probabilistic modelling*
  - Full joint probability distribution
  - Inference, complexity

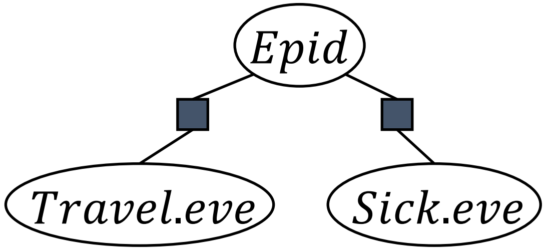
B. ***Factorised modelling***
  - (Conditional) independences
  - Factorisation

C. *Inference algorithm*
  - Variable elimination (VE)
  - Decomposition trees, complexity

Compact Encoding

- Full joint probability distribution:
  Every random variable is connected with every other random variable!
- Factorise full joint probability distribution P_F using (conditional) independences
  - Independences hidden in P_F
    - If known: explicitly represent through factors and in graph





Excursion: Multiplication

- Join over arguments + product of probabilities
  - $\phi_1(R_{11}, \dots, R_{1k}) \cdot \phi_2(R_{21}, \dots, R_{2m}) = \phi(R_1, \dots, R_l)$
    - $\{R_1, \dots, R_l\} = \{R_{11}, \dots, R_{1k}\} \cup \{R_{21}, \dots, R_{2m}\}$
  - No common arguments = cross product of ranges
  - E.g., $P(Epid, Travel.eve) \cdot P(Travel.eve, Sick.eve)$

$Epid$	$Travel.eve$	P
$false$	$false$	0.10
$false$	$true$	0.20
$true$	$false$	0.30
$true$	$true$	0.40

•

$Travel.eve$	$Sick.eve$	P
$false$	$false$	0.25
$false$	$true$	0.30
$true$	$false$	0.35
$true$	$true$	0.10

$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	$0.10 \cdot 0.25$

Multiplication

- Join over arguments + product of probabilities
  - $\phi_1(R_{11}, \dots, R_{1k}) \cdot \phi_2(R_{21}, \dots, R_{2m}) = \phi(R_1, \dots, R_l)$
    - $\{R_1, \dots, R_l\} = \{R_{11}, \dots, R_{1k}\} \cup \{R_{21}, \dots, R_{2m}\}$
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$false$	$false$	0.10
$false$	$true$	0.20
$true$	$false$	0.30
$true$	$true$	0.40

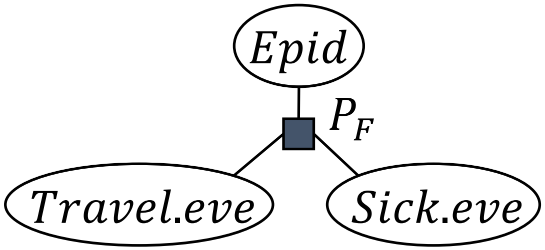
•

$Travel.eve$	$Sick.eve$	P
$false$	$false$	0.25
$false$	$true$	0.30
$true$	$false$	0.35
$true$	$true$	0.10

$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	$0.10 \cdot 0.25$
$false$	$false$	$true$	$0.10 \cdot 0.30$

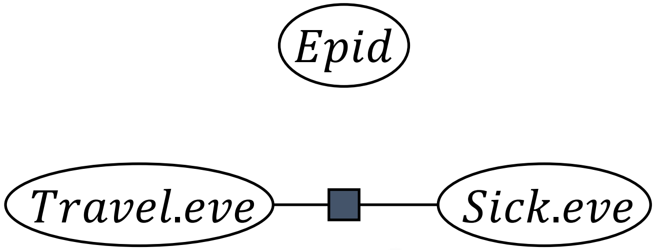
Independences

- Independence
  - $P(\boldsymbol{R},\boldsymbol{S}) = P(\boldsymbol{R}) \cdot P(\boldsymbol{S})$
  - E.g., assume we know:
    $P(Epid, Travel.eve, Sick.eve)$
    $= P(Epid)$
    $\cdot P(Travel.eve, Sick.eve)$



$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

How can we check that?



How much do we save?

Independences

- To check $P(Epid, Travel.eve, Sick.eve) = P(Epid) \cdot P(Travel.eve, Sick.eve)$
- Compute
  - $P(Epid) = \sum_{r \in \mathcal{R}(Travel.eve)} \sum_{r \in \mathcal{R}(Sick.eve)} P_F$
  - $P(Travel.eve, Sick.eve) = \sum_{r \in \mathcal{R}(Epid)} P_F$

$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

$Epid$	P
$false$	0.80
$true$	0.20

+

$Travel.eve$	$Sick.eve$	P
$false$	$false$	0.25
$false$	$true$	0.30
$true$	$false$	0.35
$true$	$true$	0.10

Independences

- To check $P(Epid, Travel.eve, Sick.eve) = P(Epid) \cdot P(Travel.eve, Sick.eve)$
- Compute
  - $P(Epid) = \sum_{r \in \mathcal{R}(Travel.eve)} \sum_{r \in \mathcal{R}(Sick.eve)} P_F$
  - $P(Travel.eve, Sick.eve) = \sum_{r \in \mathcal{R}(Epid)} P_F$

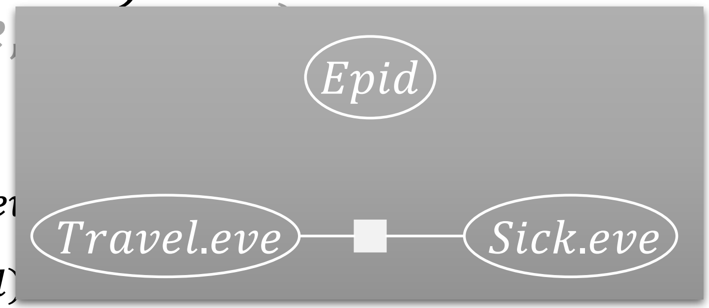
$Epid$	P
$false$	0.80
$true$	0.20

$Travel.eve$	$Sick.eve$	P
$false$	$false$	0.25
$false$	$true$	0.30
$true$	$false$	0.35
$true$	$true$	0.10

$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

Independences

- To check $P(Epid, Travel.eve, Sick.eve\ ) = P(Epid) \cdot P(Travel.eve,$
- Compute
  - $P(Epid) = \sum_{r \in \mathcal{R}(Travel.eve)} \sum_{r \in \mathcal{R}(Sick.e}$
  - $P(Travel.eve, Sick.eve) = \sum_{r \in \mathcal{R}(Epid)}$



$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

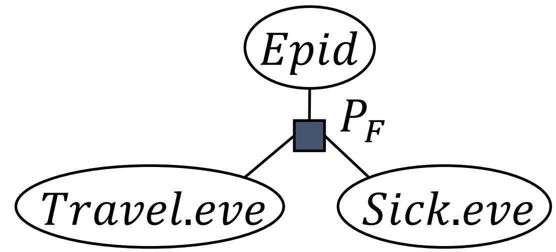
✓ =

$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

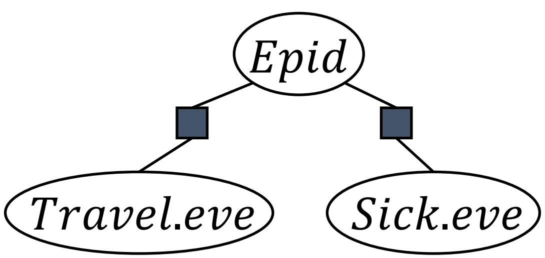
(Conditional) Independences

- Conditional independence
  - $P(\boldsymbol{R}, \boldsymbol{S} | \boldsymbol{T}) = P(\boldsymbol{R} | \boldsymbol{T}) \cdot P(\boldsymbol{S} | \boldsymbol{T})$
  - E.g.,
    $P(Travel.eve, Sick.eve | Epid)$
    $= P(Travel.eve | Epid)$
    $\cdot P(Sick.eve | Epid)$

Does it hold?



$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02



Conditional Independences

- To check $P(Travel.eve, Sick.eve|Epid)$
  $= P(Travel.eve|Epid) \cdot P(Sick.eve|Epid)$
- Replace
  - $P(Travel.eve, Sick.eve|Epid) = \frac{P_F}{P(Epid)}$
  - $P(Travel.eve|Epid) = \frac{\sum_{r \in \mathcal{R}(Sick.eve)} P_F}{P(Epid)}$
  - $P(Sick.eve|Epid) = \frac{\sum_{r \in \mathcal{R}(Travel.eve)} P_F}{P(Epid)}$
- in above equation:

$$\frac{P_F}{P(Epid)} = \frac{\sum_{r \in \mathcal{R}(Sick.eve)} P_F}{P(Epid)} \cdot \frac{\sum_{r \in \mathcal{R}(Travel.eve)} P_F}{P(Epid)}$$

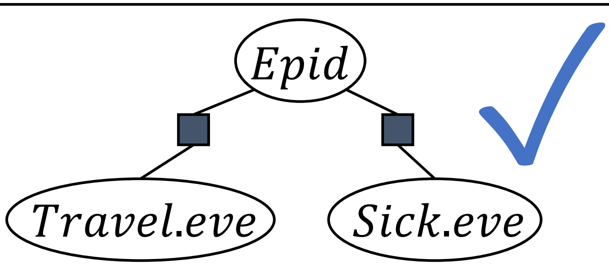
$$\Leftrightarrow P_F = \frac{\left(\sum_{r \in \mathcal{R}(Sick.eve)} P_F\right) \cdot \left(\sum_{r \in \mathcal{R}(Travel.eve)} P_F\right)}{P(Epid)}$$

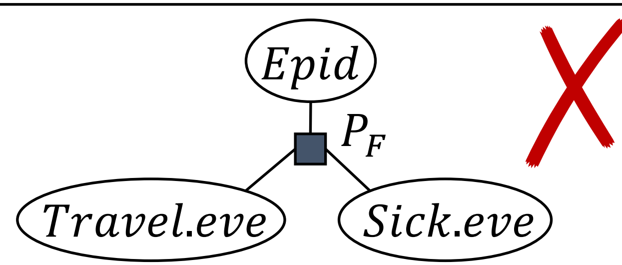
(Conditional) Independences

- $P(Travel.eve, Sick.eve|Epid)$
  $= P(Travel.eve|Epid) \cdot P(Sick.eve|Epid)$

$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.51
$false$	$false$	$true$	0.09
$false$	$true$	$false$	0.17
$false$	$true$	$true$	0.03
$true$	$false$	$false$	0.09
$true$	$false$	$true$	0.03
$true$	$true$	$false$	0.06
$true$	$true$	$true$	0.02

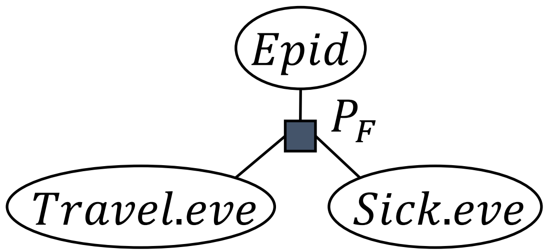
$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

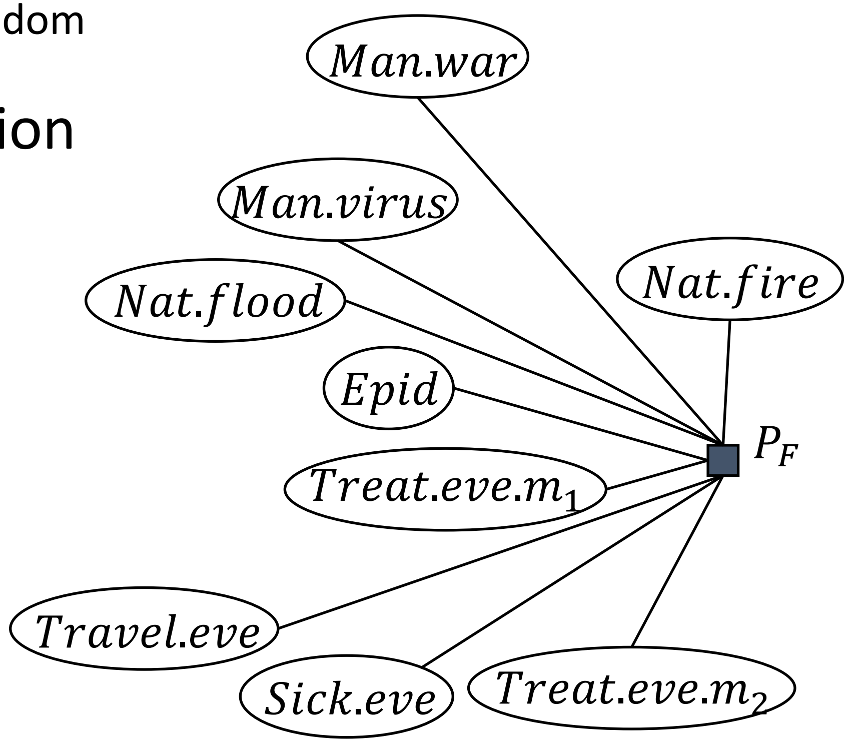




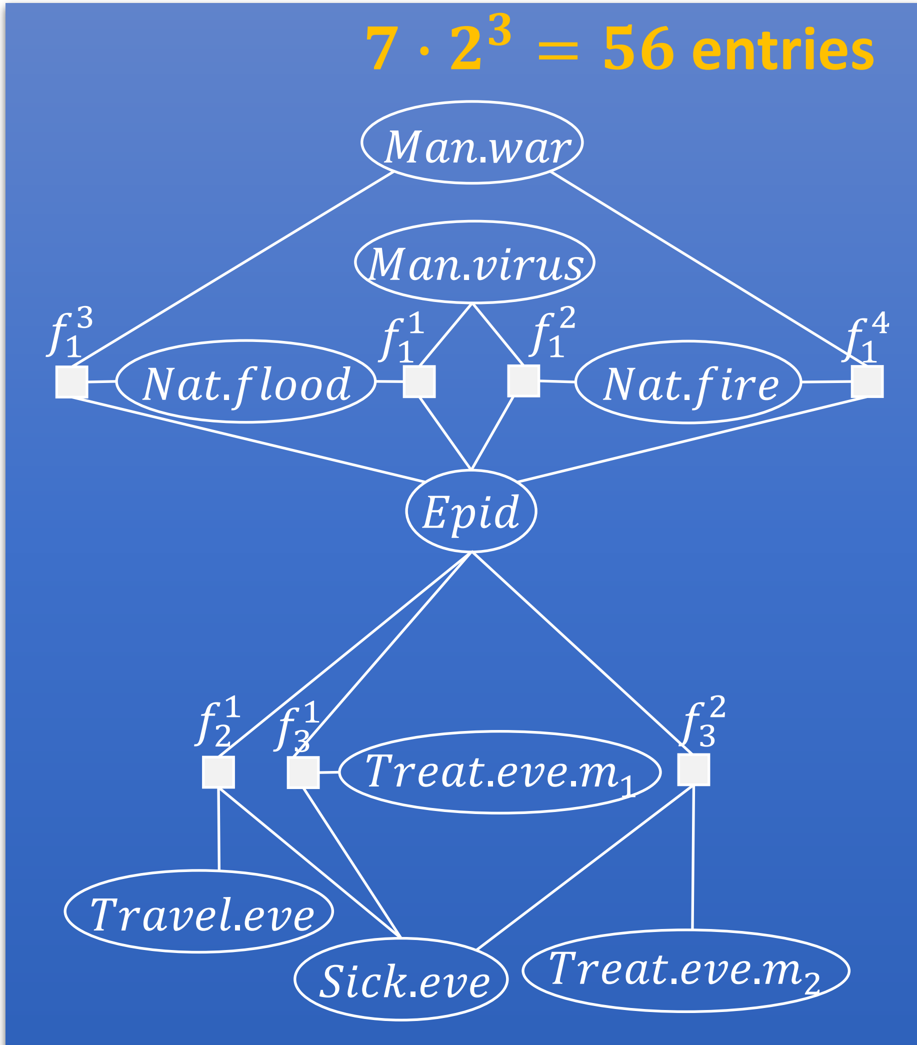
Finding a Compact Encoding

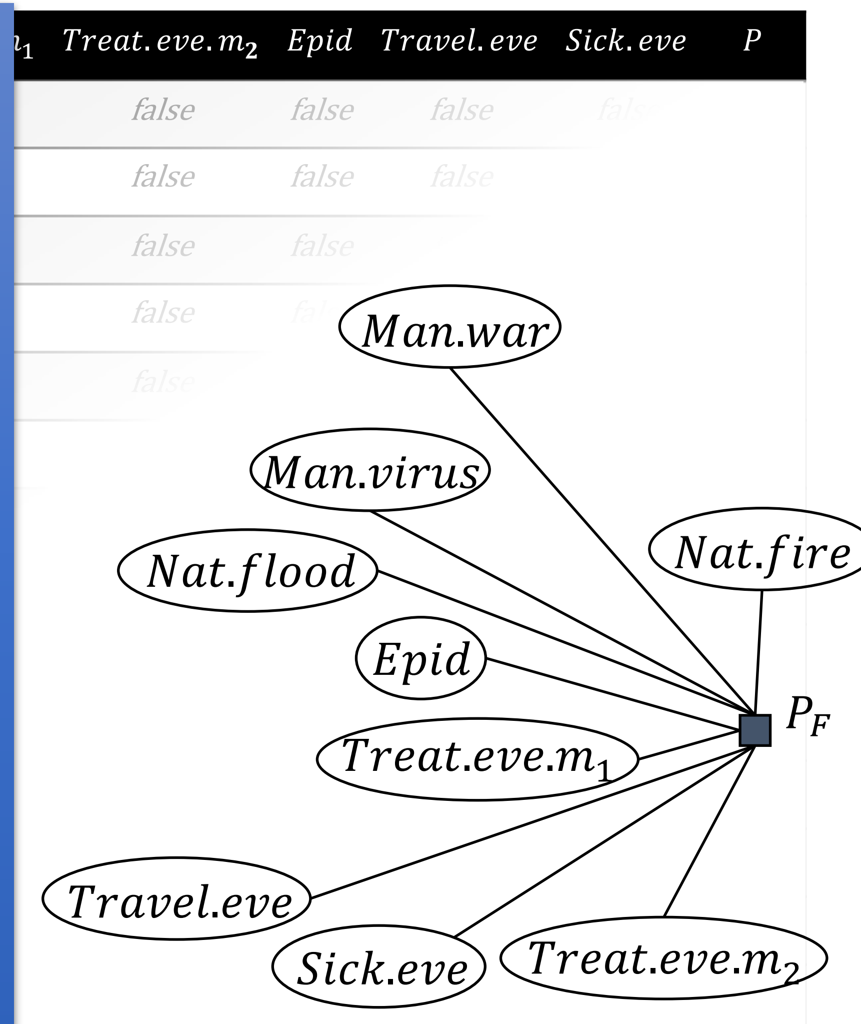
- At beginning: Everything connected with everything
- Find (conditional) independences in P_F
  - Partitions P_F into a set of factors
    - Deletes connections between random variables
- Check every possible combination → Combinatorial explosion!
  - E.g., (many more)
    - Is $Epid$ independent of $Travel.eve, Sick.eve$?
    - Is $Travel.eve$ independent of $Epid, Sick.eve$?
    - Is $Sick.eve$ independent of $Epid, Travel.eve$?
- Alternative
  - Start with no connections
  - Add factors
  - More later (→ *Section 4: Lifted Learning*)





Exponential Blowup! → Sparse Encoding!





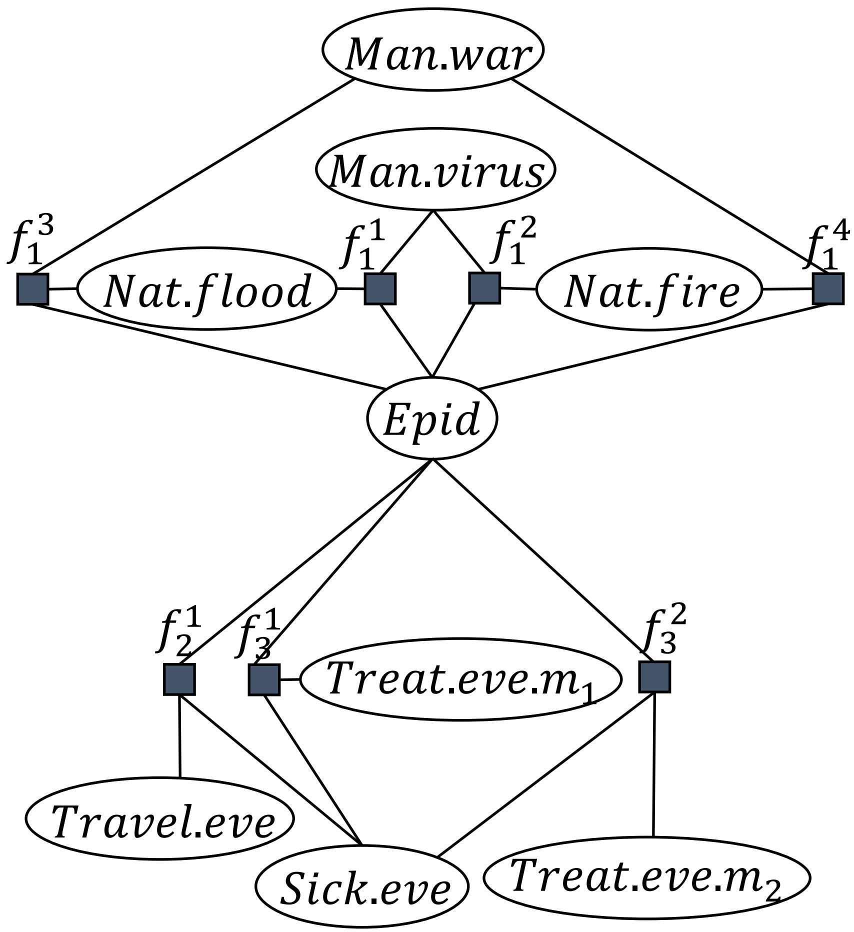
$2^9 = 512$ possible worlds

Model Representation: Factors

- Given set of random variables
  $$\boldsymbol{R} = \{R_1, \dots, R_n\}$$

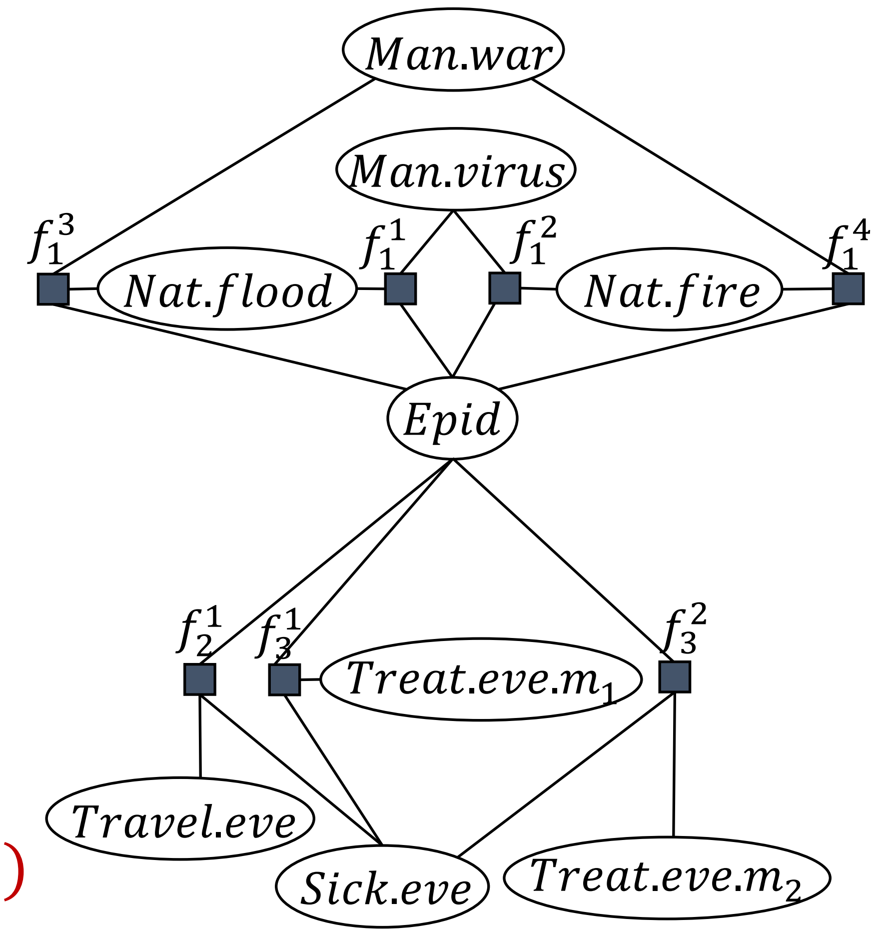
<u>Syntax</u>:

- Set of factors $F = \{f_i\}_{i=1}^n$ = model
- Factor $f = \phi(R_1, \dots, R_k)$
  - Arguments $R_1, \dots, R_k \in \boldsymbol{R}$
  - Potential function $\phi: \times_{i=1}^k \mathcal{R}(R_i) \longrightarrow \mathbb{R}^{0,+}$
    - At least one potential > 0
    - Write as table, list, …
    - Not required to be a probability distribution



Model Representation: Factors

- Given model $F = \{f_i\}_{i=1}^{n}$ over random variables
  $$\boldsymbol{R} = \{R_1, \dots, R_n\}$$
  - $f_i = \phi_i(R_1, \dots, R_k)$



<u>Semantics</u>:

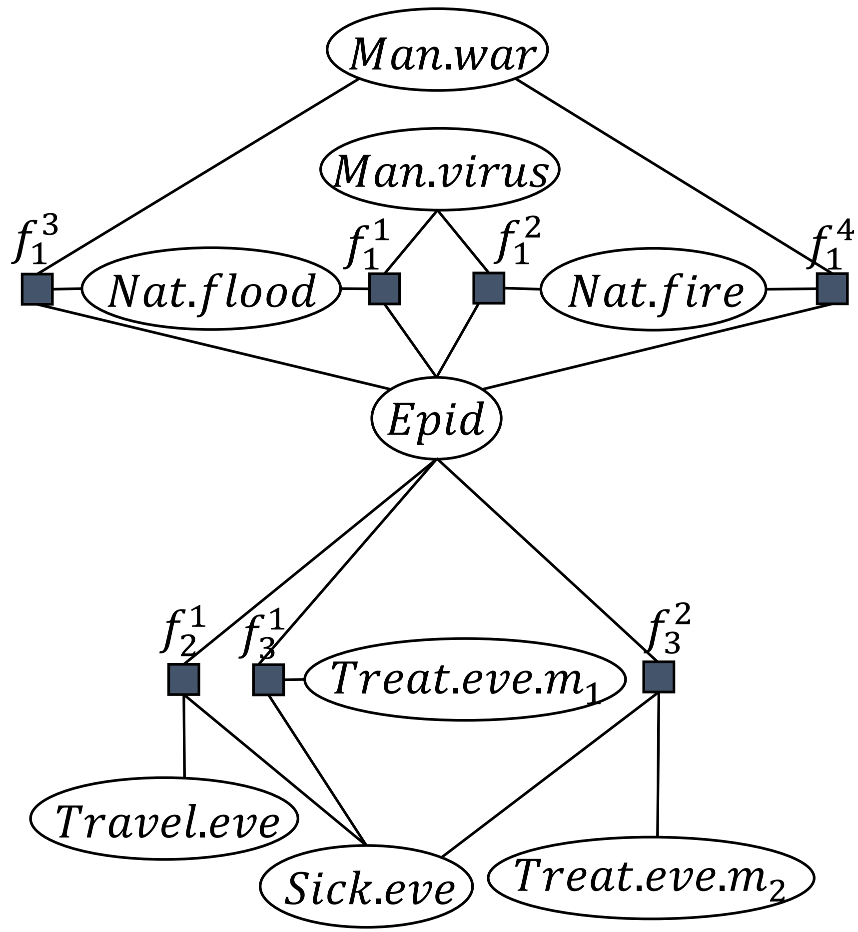
- Build full joint probability distribution P_F

$$P_F = \frac{1}{Z} \prod_{i=1}^{n} \phi_i(R_1, \dots, R_k)$$

$$Z = \sum_{r_1 \in \mathcal{R}(R_1)} \dots \sum_{r_n \in \mathcal{R}(R_n)} \prod_{i=1}^{k} \phi_i(r_1, \dots, r_k)$$

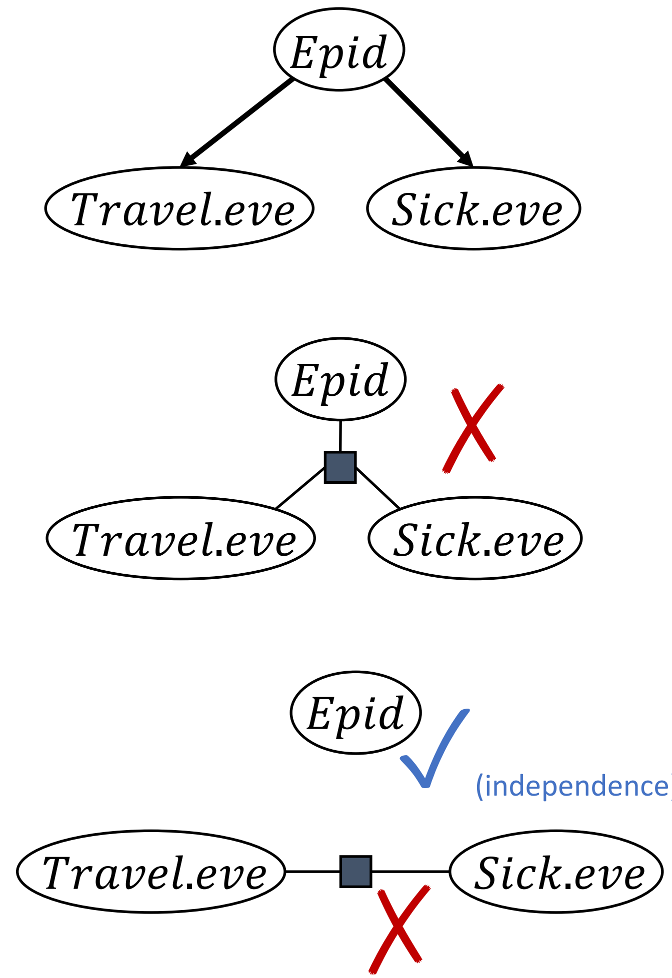
Model Representation: Factors

- Given model $F = \{f_i\}_{i=1}^n$ over random variables $\boldsymbol{R} = \{R_1, \dots, R_n\}$
  - $f_i = \phi_i(R_1, \dots, R_k)$
- Graphical representation: Factor graph (FG)
  - Each $R \in \boldsymbol{R}$: variable node in FG (ellipse)
  - Each $f \in F$: factor node in FG (box)
  - For each argument R in $f \in F$: edge between variable node for R and factor node for f



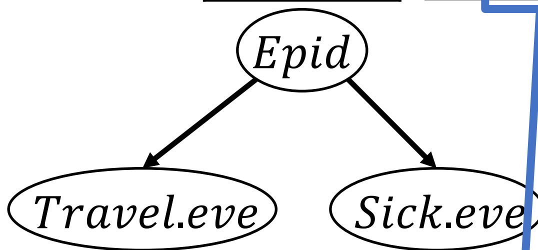
Other Model Representations

- Bayesian network
  - Directed acyclic graph
    - Cannot model bidirectional influences
      - Disadvantage!
  - Factors
    - Prior probability tables for roots
    - Conditional probability tables for all other nodes
  - Semantics
    - Product of tables
    - $Z = 1$ as tables are all probability distributions
      - Advantage!



BNs and Full Joints

$P(epid)$	$P(\neg epid)$
0.2	0.8



$Epid$	$P(travel.eve \mid Epid)$	$P(\neg t \ldots)$
$false$	0.25	0.75
$true$	0.40	0.60

$Epid$	$P(sick.eve \mid Epid)$	$P(\neg s \ldots)$
$false$	0.15	0.85
$true$	0.25	0.75

- $P_F = P(Epid) \cdot P(Travel.eve \mid Epid) \cdot P(Sick.eve \mid Epid)$
  - Multiplication as before: join over random variables, product of probabilities

Would actually be $1 - \langle stored \rangle$

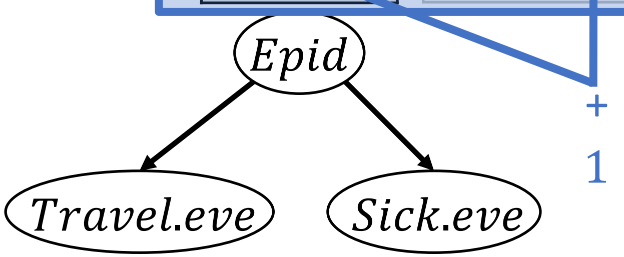
$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	$0.8 \cdot 0.75 \cdot 0.85 = 0.51$

⋮

BNs and Full Joints

Sum out Epid

$P(epid)$	$P(\neg epid)$
$0.2 \cdot 1 \cdot 1$	$0.8 \cdot 1 \cdot 1$



Sum out Travel.eve

$Epid$	$P(travel.eve \mid Epid)$	$P(\neg t \dots)$
$false$	0.25	0.75
$true$	0.40	0.60

Sum out Sick.eve

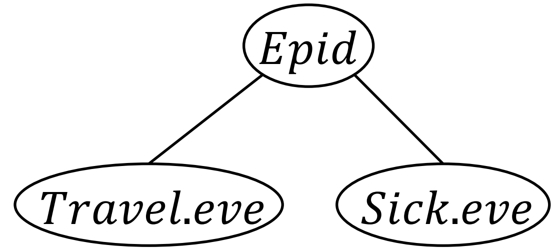
$Epid$	$P(sick.eve \mid Epid)$	$P(\neg s \dots)$
$false$	0.15	0.85
$true$	0.25	0.75

- $P_F = P(Epid) \cdot P(Travel.eve \mid Epid) \cdot P(Sick.eve \mid Epid)$
  - $Z = 1$: eliminating random variables leads to 1's
    - Also: if computing P_F, you can see, the numbers add up to 1

$Epid$	$Travel.eve$	$Sick.eve$	P
$false$	$false$	$false$	$0.8 \cdot 0.75 \cdot 0.85 = 0.51$

⋮

Other Model Representations



- Markov network
  - Undirected graph
  - Factors
    - Potential function for each clique in graph (as defined before)
  - Semantics
    - Product of all factors
    - Normalise to get full joint
- If the graph is chordal, then an equivalent BN exists
  - Chordal: all cycles of ≥ 4 nodes have a chord
    - Chord: edge not part of cycle but connects two nodes of cycle
- MN and factor model equivalent

Markov Properties

- FGs, BNs, MNs fulfil the Markov properties
  - Pairwise Markov property
    Any two non–adjacent variables are conditionally independent given all other variables
    - FG: adjacent if directly connected by factor node
  - Local Markov property
    A variable is conditionally independent of all other variables given its neighbours
    - BN: … of its non-descendants given its parent variables
  - Global Markov property
    Any two subsets of variables are conditionally independent given a separating subset
    - Separating subset S between A and B: every path from any node in A to a node in B passes through S

Markov Properties

- Hard to establish Markov properties for arbitrary probability distribution

- But: Distribution can be factorised according to cliques in graph *iff* the properties hold
  - Distribution has to be positive
    - At least one input must map to a potential > 0
    - Part of the definition of factor models/MNs

- Result of the Hammersley-Clifford Theorem

John Hammersley and Peter Clifford: Markov Fields on Finite Graphs and Lattices. In: unpublished, 1971.
Julian Besag: Spatial Interaction and the Statistical Analysis of Lattice Systems. In: *Journal of the Royal Statistical Society. Series B: Methodological*, 1974.

Interim Summary

- (Conditional) independences
  - $P(\boldsymbol{R}, \boldsymbol{S} | \boldsymbol{T}) = P(\boldsymbol{R} | \boldsymbol{T}) \cdot P(\boldsymbol{S} | \boldsymbol{T})$
  - Independence: $\boldsymbol{T} = \emptyset$
- Factorised models
  - Factor models
  - Bayesian network, Markov network (briefly)
  - Markov properties

Outline: 1. Recap: Propositional Modelling

A. *Probabilistic modelling*
   - Full joint probability distribution
   - Inference, complexity

B. *Factorised modelling*
   - (Conditional) independences
   - Factorisation

C. ***Inference algorithm***
   - Variable elimination (VE)
   - Decomposition trees, complexity

Inference Tasks

- Basically as before:
- Query Answering Problem
  - Compute an answer to a query $P(\boldsymbol{S}|\boldsymbol{T})$ given a model F representing the full joint probability distribution P_F
    - Query for a marginal (conditional) probability (distribution)

- Algorithm: Variable elimination (VE)
  - Input: Query $P(\boldsymbol{S}|\boldsymbol{T})$
  - Solving an instance of a query answering problem
    - As before: Eliminate all non-query terms
  - Algorithm for factor models, BNs, MNs, …

Nevin L. Zhang and David Poole: A Simple Approach to Bayesian Network Computations. In: *Proceedings of the 10th Canadian Conference on Artificial Intelligence*, 1994.

Variable Elimination (VE)

- Eliminate all non-query terms and normalise
  - $\boldsymbol{U} = \boldsymbol{R} \setminus rv(\boldsymbol{S}, \boldsymbol{T})$
- As before, but: Use factorisation!
  - Given $R_1 \dots R_n \in \boldsymbol{U}$:

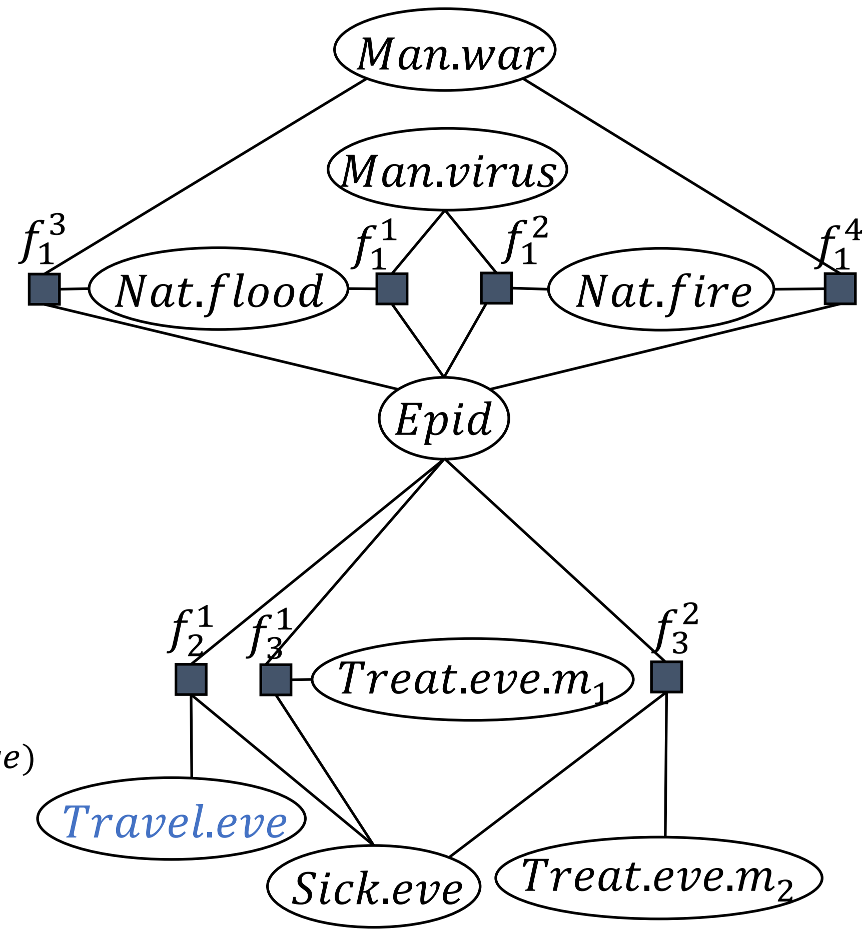
$$P(\boldsymbol{S}|\boldsymbol{T}) = \frac{1}{P(\boldsymbol{T})} \sum_{v_1 \in \mathcal{R}(R_1)} \dots \sum_{v_n \in \mathcal{R}(R_n)} P_F(R_1 = v_1, \dots, R_n = v_n, \boldsymbol{S}, \boldsymbol{T})$$

$$= \frac{1}{P(\boldsymbol{T})} \sum_{v_1 \in \mathcal{R}(R_1)} \dots \sum_{v_n \in \mathcal{R}(R_n)} \prod_{i=1}^{n} \phi_i(R_1 = v_1, \dots, R_k = v_k)$$

if $R_i \in rv(\boldsymbol{S})$ or $R_i \in rv(\boldsymbol{T})$ then
Replace $R_i = v_i$ with R_i or value of event

VE by Example

- E.g., marginal
  - $P(Travel.eve)$



$$P(Travel.eve) \propto \sum_{e \in \mathcal{R}(Epid)} \sum_{s \in \mathcal{R}(Sick.eve)} \sum_{m_1 \in \mathcal{R}(Treat.eve.m_1)} \sum_{m_2 \in \mathcal{R}(Treat.eve.m_2)} \sum_{o \in \mathcal{R}(Nat.flood)} \sum_{i \in \mathcal{R}(Nat.fire)} \sum_{w \in \mathcal{R}(Man.war)} \sum_{v \in \mathcal{R}(Man.virus)} P_F$$

VE by Example

$$P(Travel.eve) \propto \sum_{e \in \mathcal{R}(Epid)} \sum_{s \in \mathcal{R}(Sick.eve)} f_2^1(Travel.eve, e, s) \sum_{m_1 \in \mathcal{R}(Treat.eve.m_1)} f_3^1(e, s, m_1)$$

$$\sum_{m_2 \in \mathcal{R}(Treat.eve.m_2)} f_3^2(e, s, m_2)$$

$$f_1'(e)$$

$$\sum_{o \in \mathcal{R}(Nat.flood)} \sum_{i \in \mathcal{R}(Nat.fire)} \sum_{w \in \mathcal{R}(Man.war)} f_1^1(o, w, e) f_1^2(i, w, e)$$

$$f_1'(o, i, e)$$

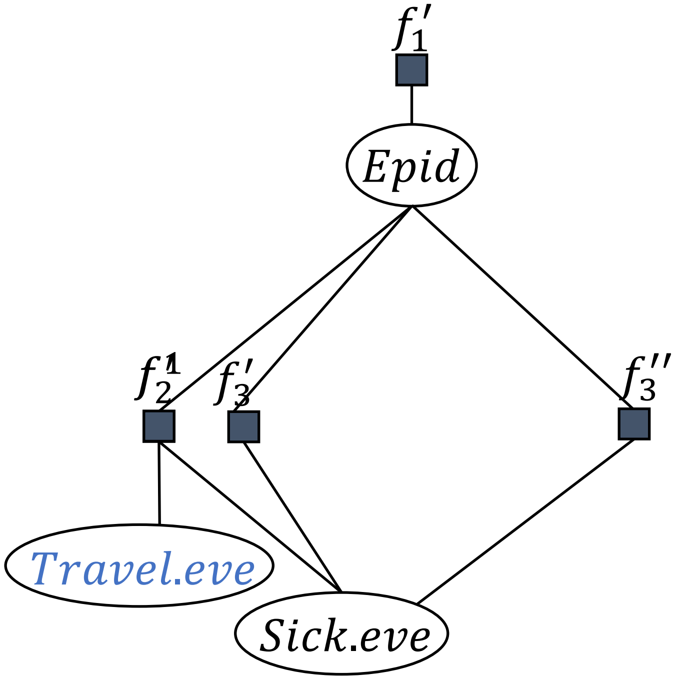
$$\sum_{\boldsymbol{v \in \mathcal{R}(Man.virus)}} \boldsymbol{f_1^3(o, v, e)} f_1^4(i, v, e)$$

VE by Example

$$P(Travel.eve) \propto \sum_{e \in \mathcal{R}(Epid)} f_1'(e) \sum_{s \in \mathcal{R}(Sick.eve)} f_2^1(Travel.eve, e, s) f_3'(e, s) f_3''(e, s)$$

$$= \sum_{e \in \mathcal{R}(Epid)} f_1'(e) \sum_{s \in \mathcal{R}(Sick.eve)} f'(Travel.eve, e, s)$$

$Travel.eve$	$Epid$	$Sick.eve$	f'
$false$	$false$	$false$	50
$false$	$false$	$true$	10
$false$	$true$	$false$	34
$false$	$true$	$true$	16
$true$	$false$	$false$	54
$true$	$false$	$true$	36
$true$	$true$	$false$	12
$true$	$true$	$true$	69

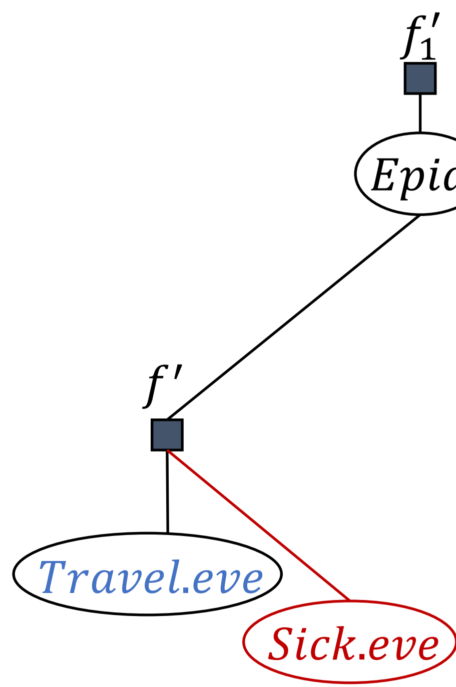


VE by Example

$$P(Travel.eve) \propto \sum_{e \in \mathcal{R}(Epid)} f_1'(e) \sum_{s \in \mathcal{R}(Sick.eve)} f_2^1(Travel.eve, e, s) f_3'(e, s) f_3''(e, s)$$

$$= \sum_{e \in \mathcal{R}(Epid)} f_1'(e) \sum_{s \in \mathcal{R}(Sick.eve)} f'(Travel.eve, e, s)$$

$Travel.eve$	$Epid$	~~$Sick.eve$~~	f'	$\sum$
$false$	$false$	~~$false$~~	50	**60**
$false$	$false$	~~$true$~~	10	
$false$	$true$	~~$false$~~	34	**50**
$false$	$true$	~~$true$~~	16	
$true$	$false$	~~$false$~~	54	**90**
$true$	$false$	~~$true$~~	36	
$true$	$true$	~~$false$~~	12	**81**
$true$	$true$	~~$true$~~	69	



VE by Example

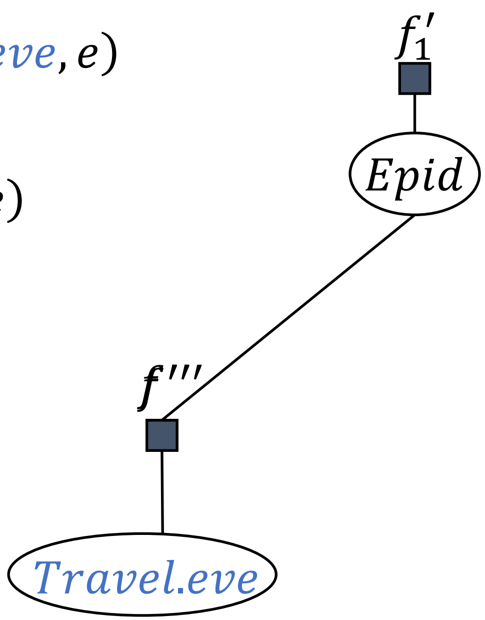
$$P(\textcolor{blue}{Travel.eve}) \propto \sum_{e \in \mathcal{R}(Epid)} f_1'(e) \sum_{s \in \mathcal{R}(Sick.eve)} f_2^1(\textcolor{blue}{Travel.eve}, e, s) f_3'(e,s) f_3''(e,s)$$

$$= \sum_{e \in \mathcal{R}(Epid)} f_1'(e) \sum_{s \in \mathcal{R}(Sick.eve)} f'(\textcolor{blue}{Travel.eve}, e, s)$$

$$= \sum_{e \in \mathcal{R}(Epid)} f_1'(e) f''(\textcolor{blue}{Travel.eve}, e)$$

$$= \sum_{e \in \mathcal{R}(Epid)} f'''(\textcolor{blue}{Travel.eve}, e)$$

$Travel.eve$	$Epid$	f'''
$false$	$false$	90
$false$	$true$	100
$true$	$false$	135
$true$	$true$	162



VE by Example

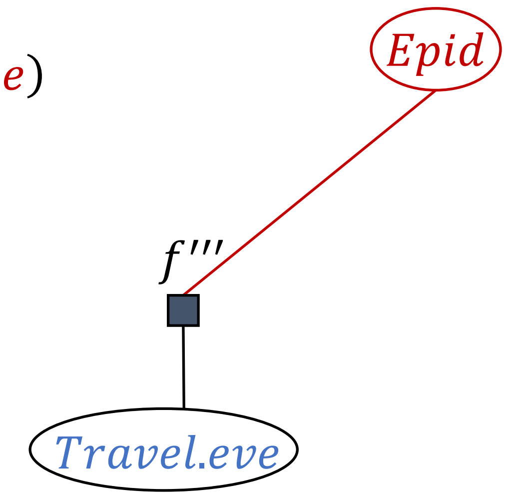
$$P(Travel.eve) \propto \sum_{e \in \mathcal{R}(Epid)} f_1'(e) \sum_{s \in \mathcal{R}(Sick.eve)} f_2^1(Travel.eve, e, s) f_3'(e, s) f_3''(e, s)$$

$$= \sum_{e \in \mathcal{R}(Epid)} f_1'(e) \sum_{s \in \mathcal{R}(Sick.eve)} f'(Travel.eve, e, s)$$

$$= \sum_{e \in \mathcal{R}(Epid)} f_1'(e) f''(Travel.eve, e)$$

$$= \sum_{e \in \mathcal{R}(Epid)} f'''(Travel.eve, e)$$

$Travel.eve$	~~$Epid$~~	f'''	Σ
$false$	~~$false$~~	90	190
$false$	~~$true$~~	100	
$true$	~~$false$~~	135	297
$true$	~~$true$~~	162	



VE by Example

$$P(Travel.eve) \propto \sum_{e \in \mathcal{R}(Epid)} f_1'(e) \sum_{s \in \mathcal{R}(Sick.eve)} f_2^1(Travel.eve, e, s) f_3'(e, s) f_3''(e, s)$$

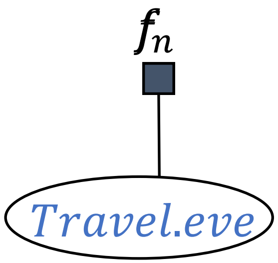
$$= \sum_{e \in \mathcal{R}(Epid)} f_1'(e) \sum_{s \in \mathcal{R}(Sick.eve)} f'(Travel.eve, e, s)$$

$$= \sum_{e \in \mathcal{R}(Epid)} f_1'(e) f''(Travel.eve, e)$$

$$= \sum_{e \in \mathcal{R}(Epid)} f'''(Travel.eve, e) = f(Travel.eve) = f_n(Travel.eve)$$

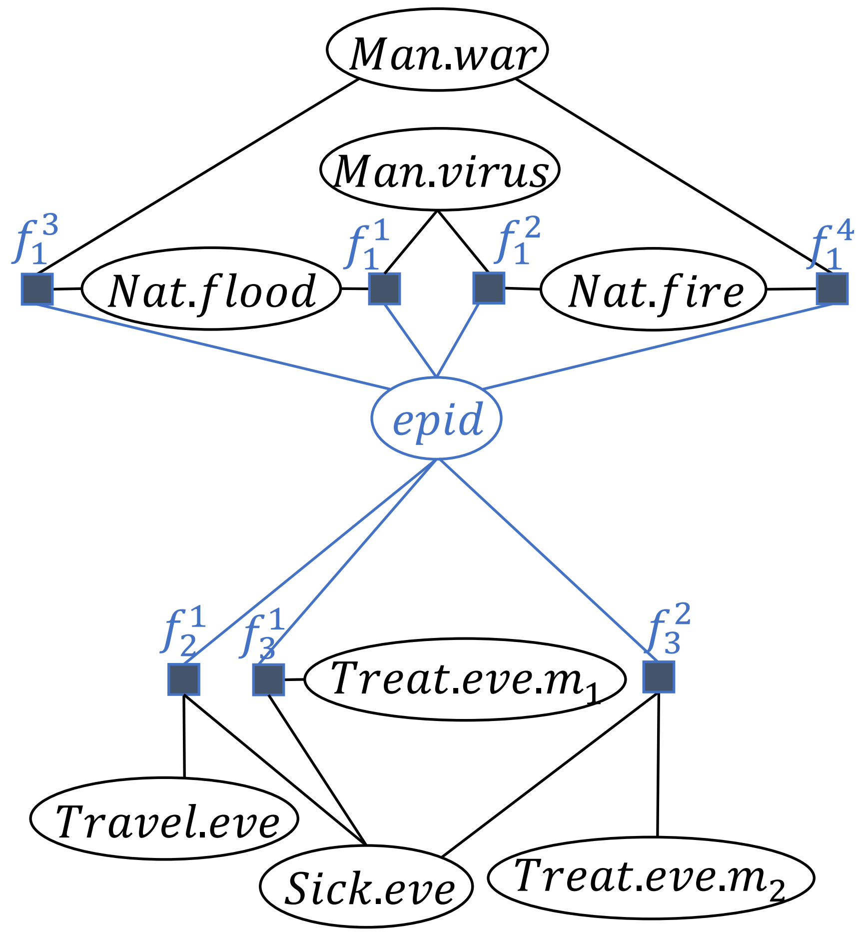
$Travel.eve$	f
$false$	190
$true$	297

$Travel.eve$	f_n
$false$	0.39
$true$	0.61



VE with Evidence

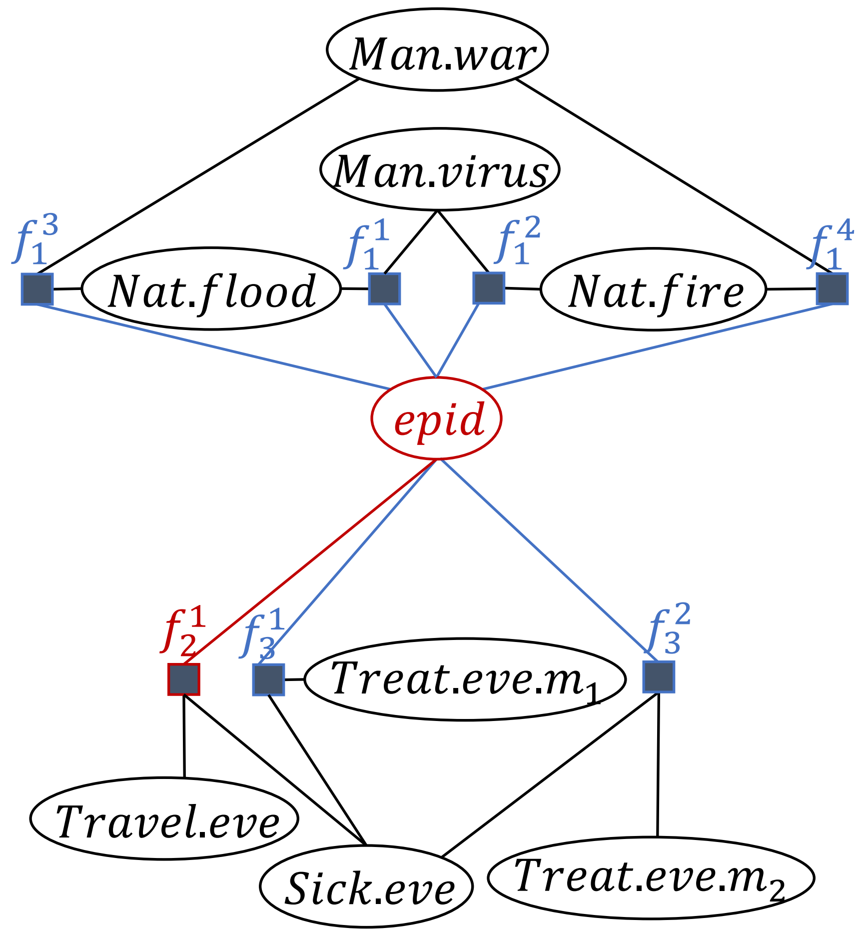
- As before
- E.g., conditional
  - $P(Travel.eve|epid)$
  - Absorb $epid$ in all factors
    - Set potentials to 0 whenever range value ≠ observed value
  - Sum out remaining variables



Evidence Absorption

- Absorb *epid* in all factors, e.g., f_2^1
- Drop lines with zeros

$Travel.eve$	$Epid$	$Sick.eve$	f_2^1
$false$	$false$	$false$	~~5~~ 0
$false$	$false$	$true$	~~0~~ 0
$false$	$true$	$false$	4
$false$	$true$	$true$	6
$true$	$false$	$false$	~~4~~ 0
$true$	$false$	$true$	~~6~~ 0
$true$	$true$	$false$	2
$true$	$true$	$true$	9

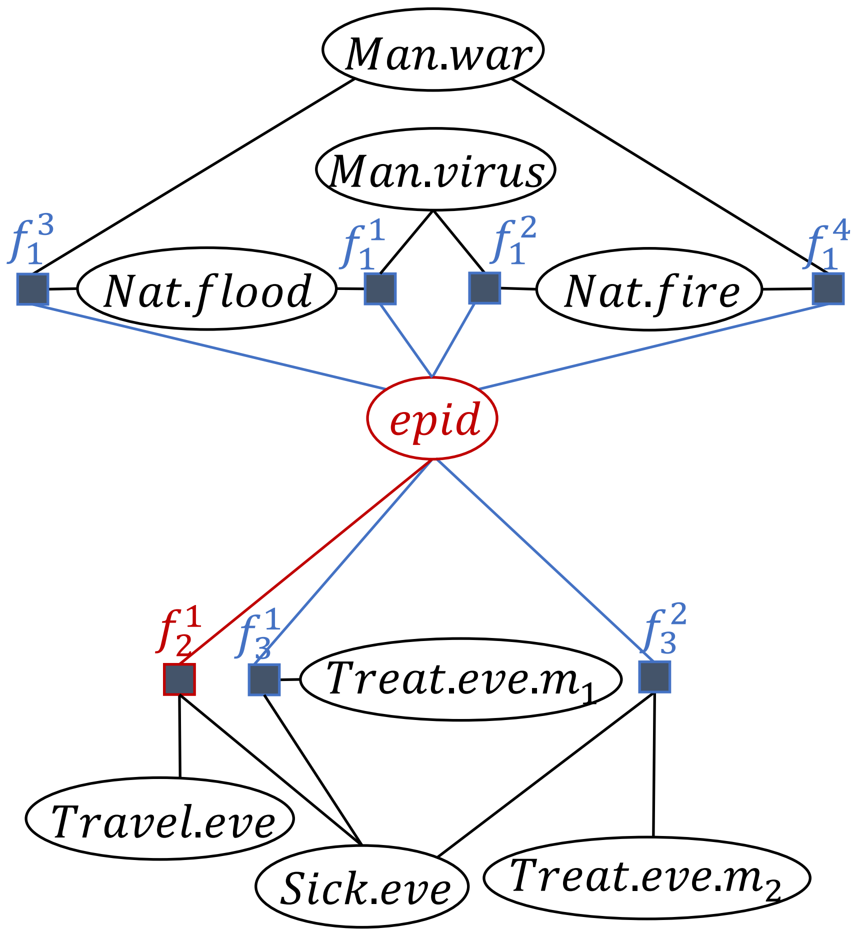


Evidence Absorption

- Eliminate variable

$Travel.eve$	$Epid$	$Sick.eve$	f_2^1
$false$	$true$	$false$	4
$false$	$true$	$true$	6
$true$	$true$	$false$	2
$true$	$true$	$true$	9

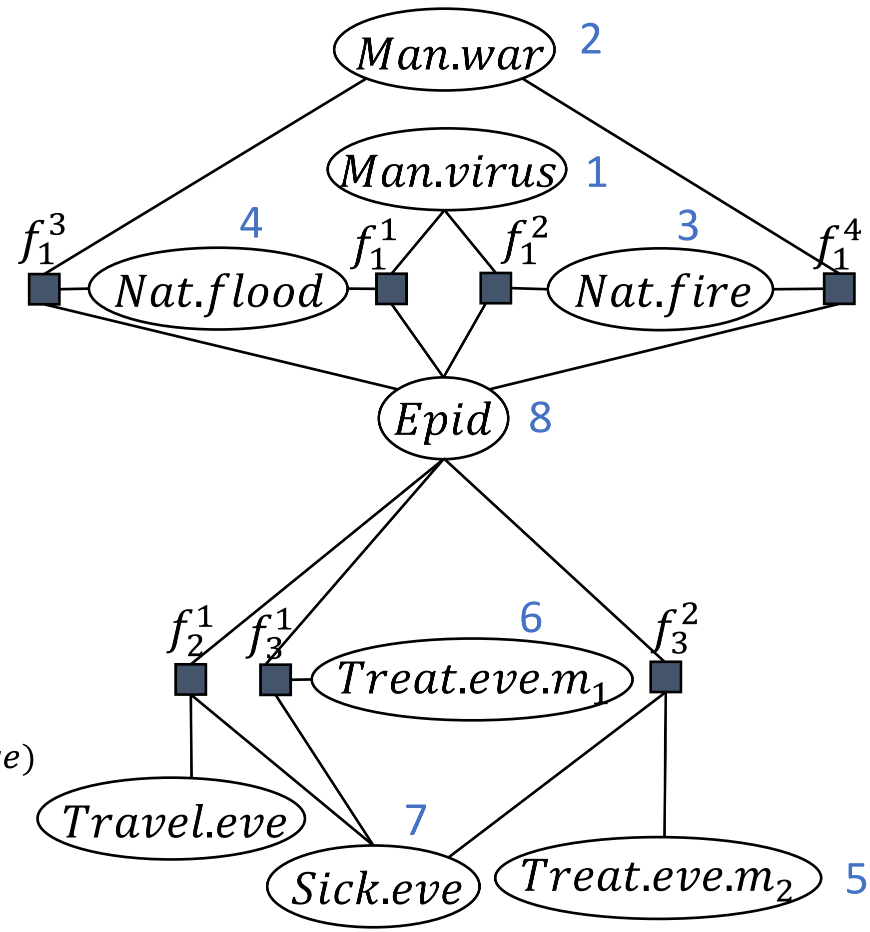
$Travel.eve$	$Sick.eve$	f_2^1
$false$	$false$	4
$false$	$true$	6
$true$	$false$	2
$true$	$true$	9



- Eliminate remaining variables…

Elimination Order & Efficiency

- Order in which random variables are eliminated = elimination order
  - E.g., (see # at nodes)

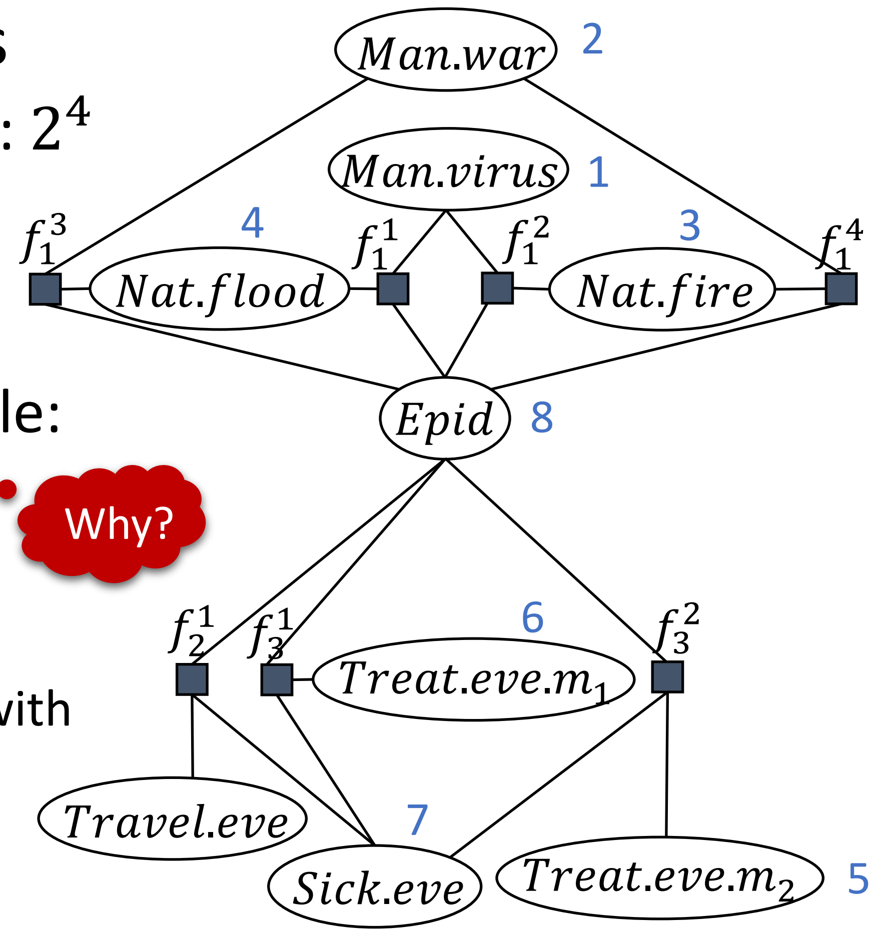


$$P(Travel.eve) \propto \sum_{e \in \mathcal{R}(Epid)} \sum_{s \in \mathcal{R}(Sick.eve)} \sum_{m_1 \in \mathcal{R}(Treat.eve.m_1)} \sum_{m_2 \in \mathcal{R}(Treat.eve.m_2)} \sum_{o \in \mathcal{R}(Nat.flood)} \sum_{i \in \mathcal{R}(Nat.fire)} \sum_{w \in \mathcal{R}(Man.war)} \sum_{v \in \mathcal{R}(Man.virus)} P_F$$

Elimination Order & Efficiency

- Quality of order:
  Size of intermediate results
  - E.g., with order on the right: 2^4
    - $f_1^1 \cdot f_1^2$ to eliminate $Man.virus$: 4 arguments
    - Smallest size possible
  - Worst case order for example: any order with $Epid$ first

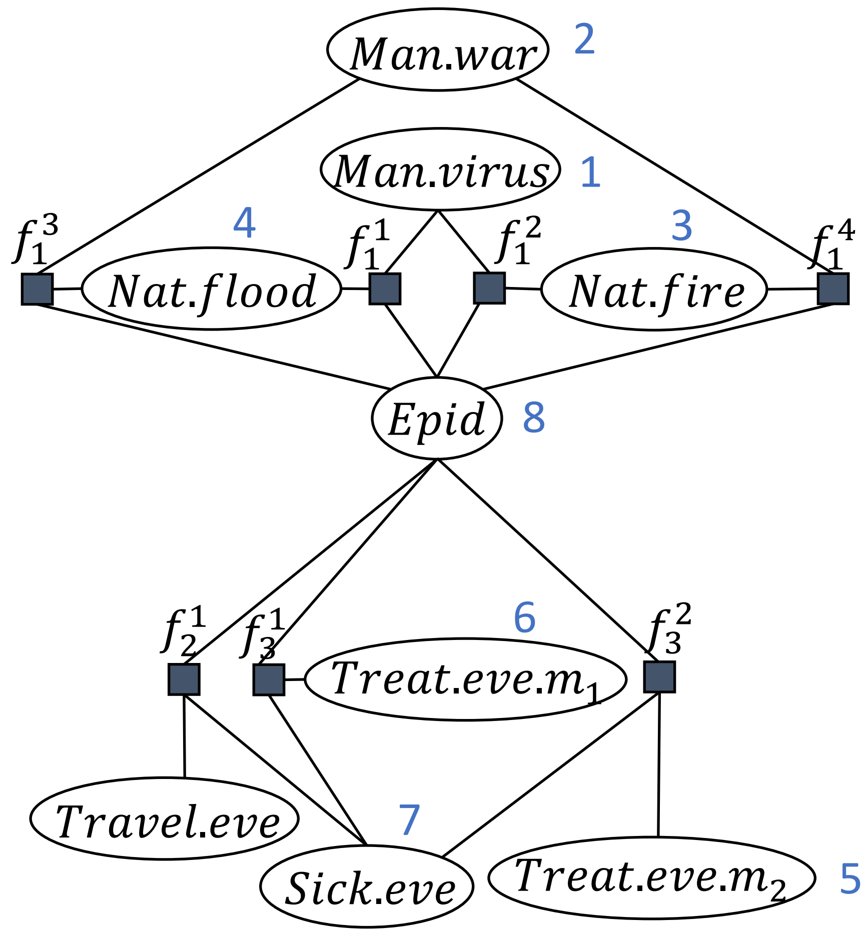
    Why?
  - Order with highest quality not necessarily unique
    - Example has several orders with size 2^4



Elimination Order & Efficiency

- Finding the order with the minimal maximum size of intermediate results: NP-hard problem
  - One possible heuristics
    - Minimum degree: Eliminate variable which results in smallest new intermediate factor

→ minimal maximum size important for *complexity*

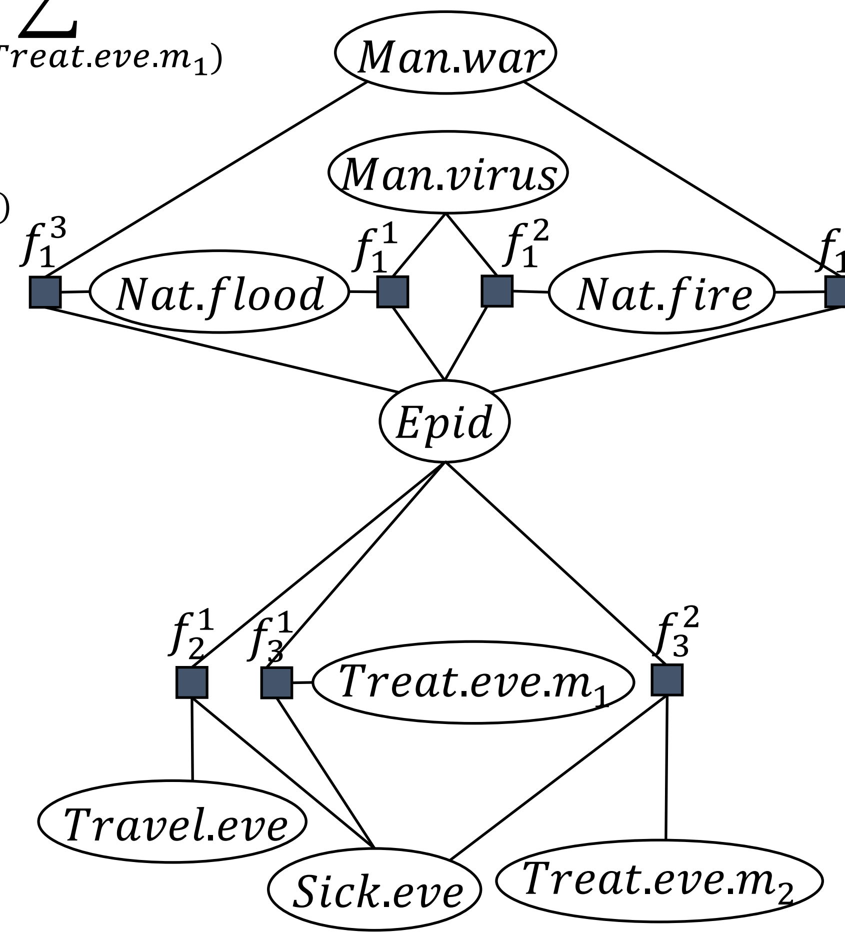


Complexity

- *Informal*
  worst case size of an (intermediate) factor
    $times$ # of eliminations
- Decomposition tree
  - Needed for formal definition
  - Representation of VE computation, e.g., for computing Z
  - Acyclic tree
    - Leaves: Factors
    - Inner nodes: Intermediate factors
      - Root: result
    - Edges: If (factors are multiplied and subsequently) a random variable is eliminated from the (intermediate) factor resulting in a new intermediate factor (inner node), then there is an edge from the factor(s) before multiplication to the inner node of the intermediate factor

Example: Computing Z*

$$Z \propto \sum_{e \in \mathcal{R}(Epid)} \sum_{s \in \mathcal{R}(Sick.eve)} \sum_{t \in \mathcal{R}(Travel.eve)} \sum_{m_1 \in \mathcal{R}(Treat.eve.m_1)} \sum_{m_2 \in \mathcal{R}(Treat.eve.m_2)} \sum_{o \in \mathcal{R}(Nat.flood)} \sum_{i \in \mathcal{R}(Nat.fire)} \sum_{w \in \mathcal{R}(Man.war)} \sum_{v \in \mathcal{R}(Man.virus)} P_F$$



*(also known as the empty query $P(\,)$)

Bottom-up Construction

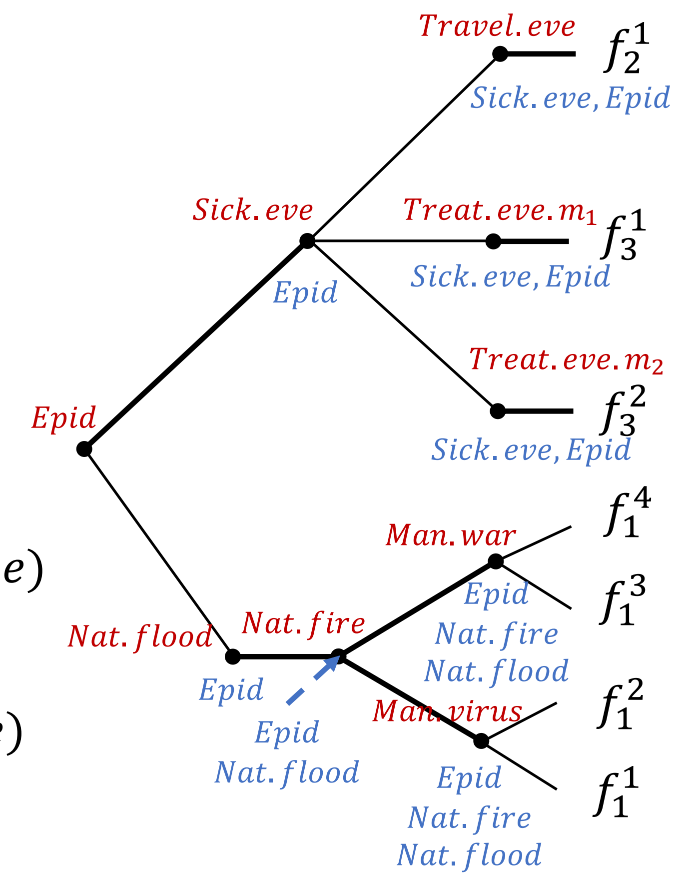
$$Z \propto \sum_{e \in \mathcal{R}(Epid)} \sum_{s \in \mathcal{R}(Sick.eve)} \sum_{t \in \mathcal{R}(Travel.eve)} f_2^1(t,e,s)$$

$$\sum_{m_1 \in \mathcal{R}(Treat.eve.m_1)} f_3^1(e,s,m_1)$$

$$\sum_{m_2 \in \mathcal{R}(Treat.eve.m_2)} f_3^2(e,s,m_2)$$

$$\sum_{o \in \mathcal{R}(Nat.flood)} \sum_{i \in \mathcal{R}(Nat.fire)} \sum_{w \in \mathcal{R}(Man.war)} f_1^3(o,w,e) f_1^4(i,w,e)$$

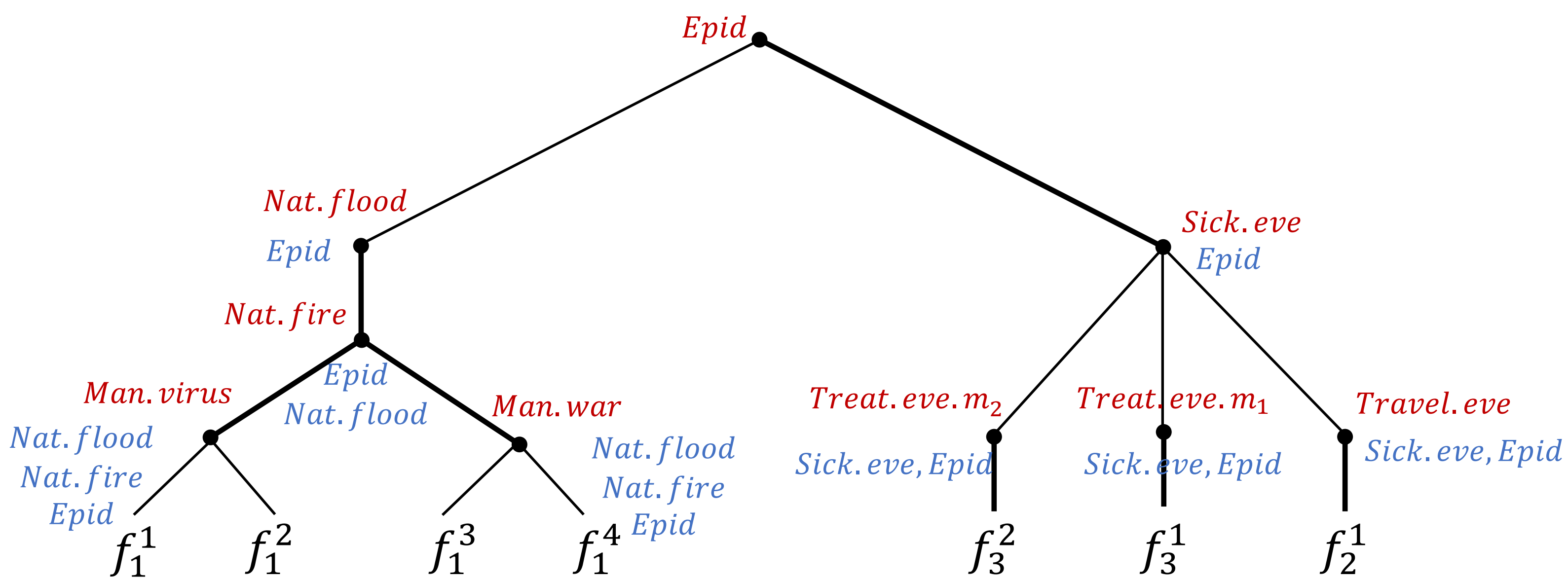
$$\sum_{v \in \mathcal{R}(Man.virus)} f_1^1(o,v,e) f_1^2(i,v,e)$$



- Computations from different branches can be carried out concurrently

Top-down Interpretation

- At start: $F = \{f_i\}_{i=1}^{n}$ at root
- Recursively find partitions of current F s.t.
  - Each partition F_i contains as many random variables $\boldsymbol{U}_i$ as possible that do not occur in any other partition while sharing as few random variables as possible
    - $\boldsymbol{U}_i$ can be eliminated without considering the factors containing $\boldsymbol{U}_j, j \neq i$
  - Add each partition as child node *i* with current model F_i
  - Until only one factor per current F



Cutset, Context, Cluster

- Cutset

$$cutset(T) = \left(\bigcup_{T_i, T_j \in children(T)} rv(T_i) \cap rv(T_j) \right) \setminus acutset(T)$$

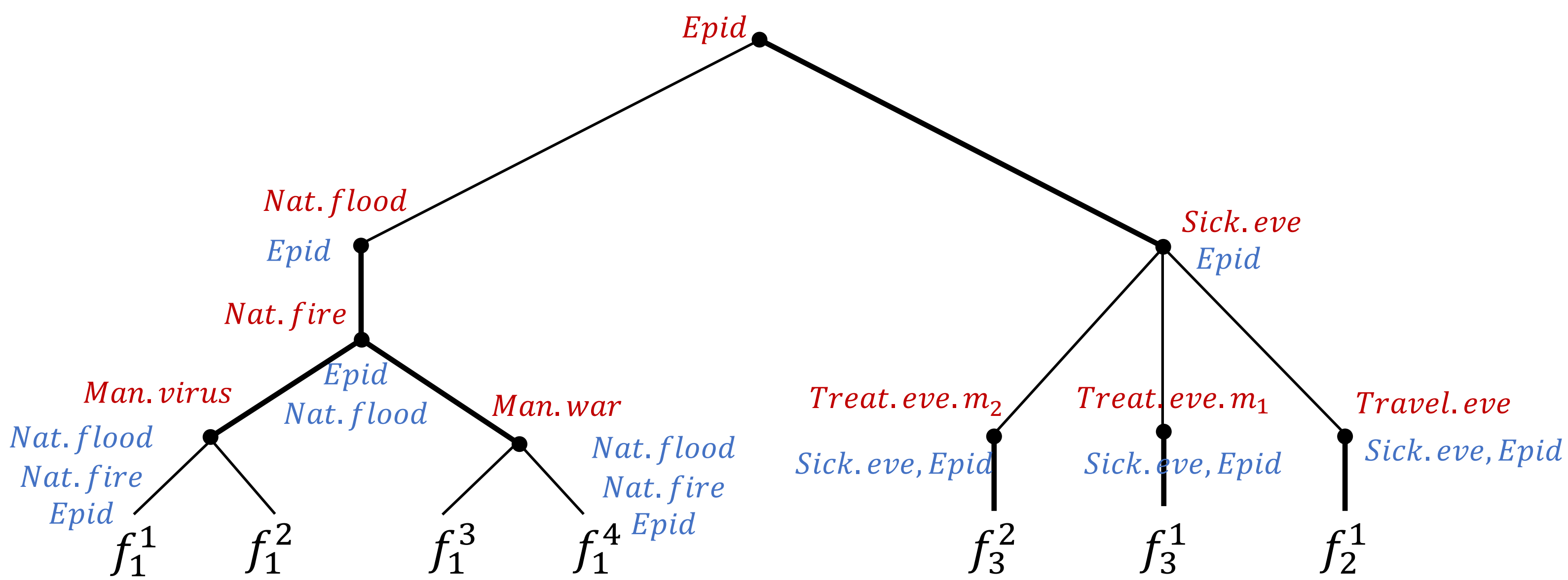
$$acutset(T) = \bigcup_{T' \in ancestor(T)} cutset(T')$$

- Context

$$context(T) = rv(T) \cap acutset(T)$$

- Cluster

$$cluster(T) = cutset(T) \cup context(T)$$

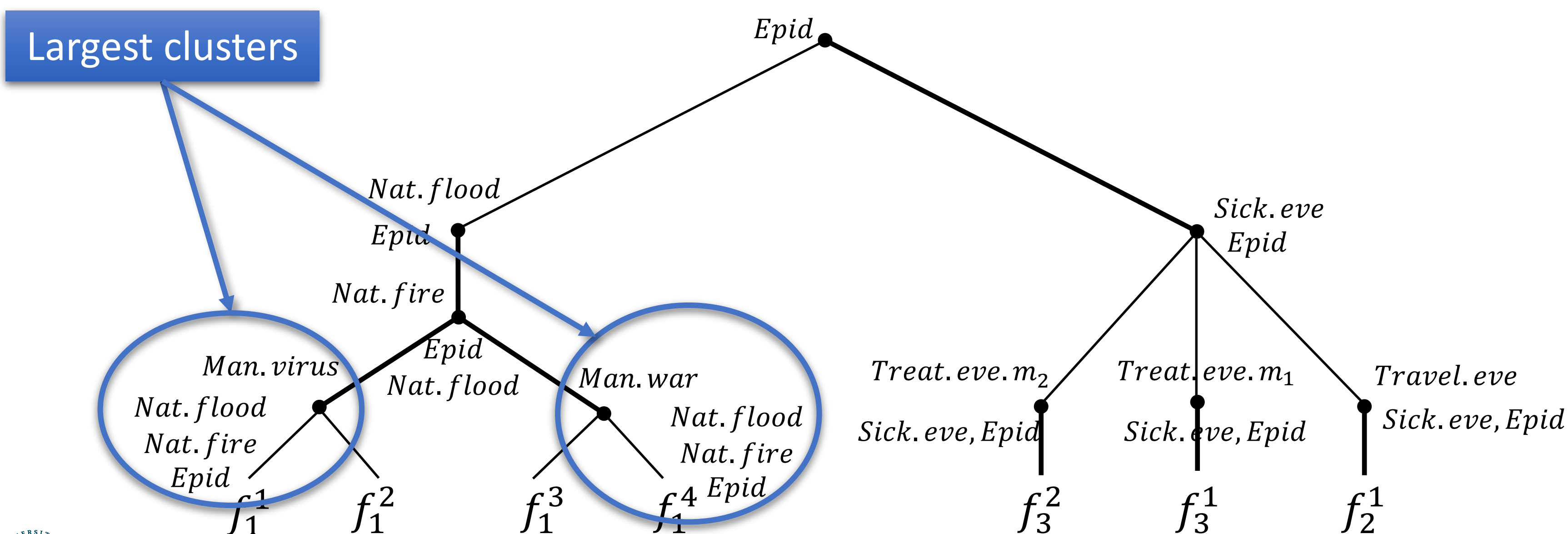


Cutset, Context, Cluster

- Largest cluster size of tree T
  = tree width w

$$w = \max_{T_i \in inner-nodes(T)} |cluster(T_i)|$$

- Induces a worst case factor size
- In example: 4, worst-case size 2^4



Back to Complexity

- *Informal*
  worst case size of an (intermediate) factor
  $times$ # of eliminations
- Decomposition tree T
  - Tree width w
    = worst case size of an (intermediate) factor
  - Number of inner nodes n_T
    = # of eliminations
- *Formal:* Runtime complexity of VE

$$O(n_T \cdot r^w)$$

  - $r = \max_{R \in \boldsymbol{R}} |\mathcal{R}(R)|$

Complexity & Tractability

- Runtime complexity of VE

$$O(n_T \cdot r^w)$$

  - Inference using P_F: $O(r^{n_T}), n_T = n = |\boldsymbol{R}|$
    - Hopefully, $w \ll n$
  - w bounded from below by maximum number of arguments among the factors in F:

$$w \geq \max_{\phi(R_1,\ldots,R_k)\in F} k$$

    - When constructing or learning, avoid high degree nodes/factors with many arguments
- A query answering problem is tractable
  - if it is solved by an efficient algorithm, running in time polynomial in the number of random variables
- Query answering problem in general intractable
  - Unless you make assumptions, e.g., about the model structure

Other Inference Algorithms

- Propagation algorithms
  - In polytree BNs: probability propagation along the edges
    Judea Pearl: Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference. Morgan-Kaufmann, 1988.
    - Polytree: the underlying *undirected* graph of a BN is acyclic
  - Generalisation: junction tree algorithm
    Steffen L. Lauritzen and David J. Spiegelhalter: Local Computations with Probabilities on Graphical Structures and Their Application to Expert Systems. In: *Journal of the Royal Statistical Society. Series B: Methodological*, 1988.
    Glenn R. Shafer and Prakash P. Shenoy: Probability Propagation. In: *Annals of Mathematics and Artificial Intelligence*, 1989.
    Finn V. Jensen, Steffen L. Lauritzen, and Kristian G. Olesen: Bayesian Updating in Recursive Graphical Models by Local Computations. In: *Computational Statistics Quarterly*, 1990.
    - Uses a helper structure based on clusters
    - Also known as join tree or clique tree
  - Knowledge compilation
    Adnan Darwiche: A Differential Approach to Inference in Bayesian Networks. In: *Proceedings of the 16th Conference on Uncertainty in Artificial Intelligence*, 2000.
    - Build a helper structure called a circuit that encodes computing Z
- Approximate algorithms…
- More in Section 3: *Lifted Inference*

Interim Summery

- Inference task: solving a query answering problem
  - As before
- Variable elimination
  - Use factorisation
  - Complexity
    - Decomposition trees
      - Cutset, context, cluster
      - Treewidth
    - Tractability

Outline: 1. Recap: Propositional Modelling

A. *Probabilistic modelling*
   - Full joint probability distribution
   - Inference, complexity

B. *Factorised modelling*
   - (Conditional) independences
   - Factorisation

C. *Inference algorithm*
   - Variable elimination (VE)
   - Decomposition trees, complexity

⟹ Next: Probabilistic Relational Models