UNIVERSITÄT ZU LÜBECK
INSTITUT FÜR INFORMATIONSSYSTEME

Dynamic Probabilistic Relational Models

$$\neg(p \wedge (q \Rightarrow r))$$
$$\neg(p \wedge (\neg q \vee r))$$
$$\neg p \vee \neg(\neg q \vee r)$$
$$\neg p \vee (\neg\neg q \wedge \neg r)$$
$$\neg p \vee (q \wedge \neg r)$$

Foundations: Logic

$$knows(jack, jill)$$
$$\wedge$$
$$\forall X, Y, X \in \mathrm{D} \wedge Y \in \mathrm{D}.$$
$$(knows(X, Y) \Rightarrow knows(Y, X))$$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

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Contents

Overview: 2. Foundations

A. Logic

- Propositional logic: alphabet, grammar, normal forms, rules
- First-order logic: introducing quantifiers, domain constraints

B. Probability theory

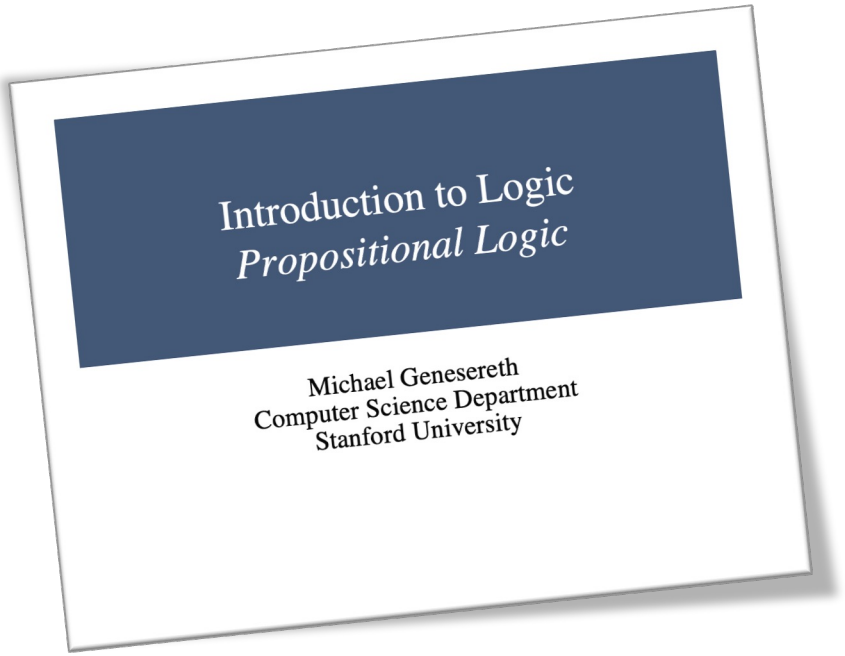
- Modelling: (conditional) probability distributions, random variables, marginal and joint distributions
- Inference: axioms and basic rules, Bayes theorem, independence

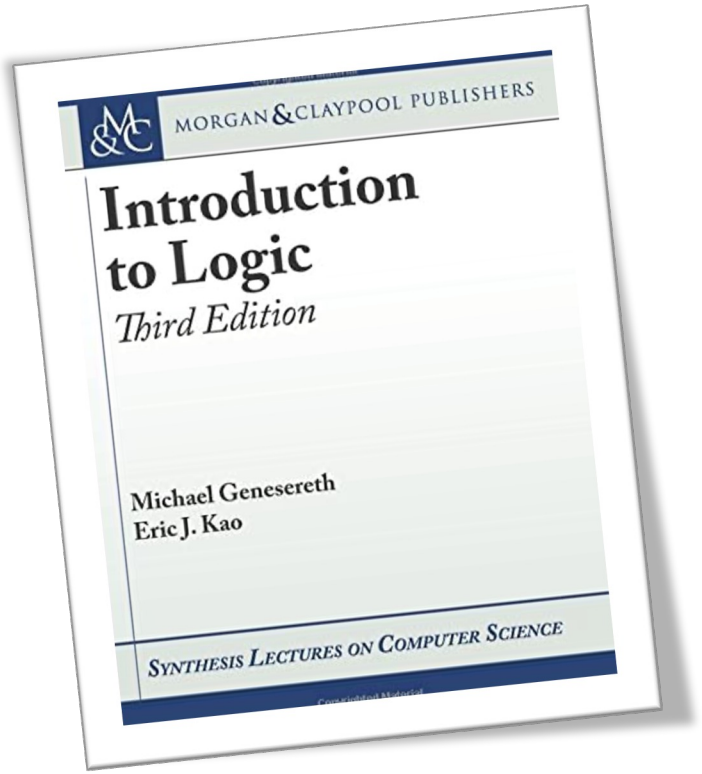
C. Probabilistic graphical models

- Syntax, semantics
- Inference problems

Sources

- Slides based on the lecture “Introduction to Logic” by Michael Genesereth at Stanford University, which has a book: “Introduction to Logic” by Michael Genesereth & Eric J. Kao





Logic

- Logic for natural language modelling, hardware + software engineering, games etc.
- Describe constraints about a world
- Components
  - Logical language

Dana likes Cody. *Abby does **not** like Dana.* *Abby likes **everyone** that Bess likes.*

  - Logical reasoning

*Dana does **not** like Abby.* *Bess likes Cody **or** Dana.*

Premises:
Dana likes Cody.
Abby does not like Dana.
Everybody likes somebody.
Bess likes Cody or Dana.
Abby likes everyone that Bess likes.
Cody likes everyone who likes her.
Nobody likes herself.

Truths:
Bess likes Cody.
Bess does not like Dana.
Everybody likes someone.

Falsehoods:
Bess likes Dana.
Everybody likes everybody.

Unknowns:
Dana likes Bess.

- Rules of inference
- Soundness and completeness
- Model checking (enumeration of cases)

Syntax

- Propositional vocabulary:
  - Set of of primitive symbols called *proposition constants*
  - Convention: strings of alphanumeric characters, starting lowercase
    - $raining, umbrella, sick, parent$
    - p, q, r
    - $parent1, parent2, r32aining, rAiNiNg$
- Wrong: $4815162342, rain. snowing$

- Propositional sentence
  - Member of propositional vocabulary
  - Compound expression: combination of
    - Member of vocabulary
    - Logical operators (¬,∧,∨, ⇒, ⇔)
    - Parentheses
  - Subexpressions sometimes referred to using Greek lowercase letters: ϕ, ψ
- Propositional language
  - Set of all propositional sentences that can be formed from a propositional vocabulary

Syntax

- Compound sentences
  - Negation: $\neg raining$
    - Argument called *target*
  - Conjunction: $raining \wedge umbrella$
    - Arguments called *conjuncts*
  - Disjunction: $raining \vee umbrella$
    - Arguments called *disjuncts*
  - Implication: $raining \Rightarrow umbrella$
  - Equivalence: $raining \Leftrightarrow umbrella$
  - Further nested sentences
    - $\neg\, raining \vee umbrella$
    - $\neg\, (raining \wedge umbrella)$

- Precedence: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
  - Allows for omitting parentheses
    - $\neg p \vee q \equiv \big((\neg p) \vee q\big)$
    - $p \vee q \wedge r \equiv \big(p \vee (q \wedge r)\big)$
    - $p \vee q \Rightarrow r \equiv \big((p \vee q) \Rightarrow r\big)$
    - $p \Rightarrow q \Leftrightarrow r \equiv \big((p \Rightarrow q) \Leftrightarrow r\big)$
  - Association
    - $\wedge, \vee$ *left-associative*
      - $p \wedge q \wedge r \equiv \big((p \wedge q) \wedge r\big)$
    - $\Rightarrow, \Leftrightarrow$ *right-associative*
      - $p \Rightarrow q \Rightarrow r \equiv \big(p \Rightarrow (q \Rightarrow r)\big)$

Semantics

- Propositional interpretation (*possible world*)
  - Association between propositional constants in propositional language and *truth values* T or F

$$p \xrightarrow{i} \mathrm{T} \qquad p^i = \mathrm{T}$$
$$q \xrightarrow{i} \mathrm{F} \qquad q^i = \mathrm{F}$$
$$r \xrightarrow{i} \mathrm{T} \qquad r^i = \mathrm{T}$$

  - Sometimes considered a Boolean vector of values for items in the ordered signature of the language
    - Ordered signature: constants of the language in a given order
    - Order: pqr
    - Interpretation $i = \mathrm{TFT}$
  - Sometimes T or F denoted by 1,0

Semantics

One interpretation is as good as any other in absence of additional information.

- Sentential interpretation
  - Association between sentences in a propositional language and truth values T or F
  - Propositional interpretation defines sentential interpretation by applying operator semantics
    - *Operator semantics*

ϕ	$\neg\phi$
T	F
F	T

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

ϕ	ψ	$\phi \Leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

  - *Evaluation :*

$$p^i = \mathrm{T} \qquad (p \vee q)^i = (\mathrm{T} \vee \mathrm{F}) = \mathrm{T}$$
$$q^i = \mathrm{F} \qquad (\neg q \vee r)^i = (\neg \mathrm{F} \vee \mathrm{T}) = (\mathrm{T} \vee \mathrm{T}) = \mathrm{T}$$
$$r^i = \mathrm{T} \qquad \big((p \vee q) \wedge (\neg q \vee r)\big)^i = (\mathrm{T} \wedge \mathrm{T}) = \mathrm{T}$$

Semantics

- Truth table
  - Table of all possible interpretations for the propositional constants in a language
    - One column per constant
    - One row per interpretation
    - For a language with n constants, there are 2^n interpretations
      - Exponential!

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Satisfaction (Model Checking)

- Find all propositional interpretations that satisfy a given set of sentences
  - Satisfy: each sentence evaluates to T
  - Satisfying interpretation then often called *model*
  - Procedure
    1. Form a truth table for propositional constants
    2. For each sentence in the set and each row in the truth table, check whether the row satisfies the sentence
       - Cross out rows that do not
       - Result: remaining rows, which satisfy all sentences in the given set
  - Example
    - $q \Rightarrow r$
    - $p \Rightarrow q \wedge r$
    - $\neg r$

p	q	r
~~T~~	~~T~~	~~T~~
~~T~~	~~T~~	~~F~~
~~T~~	~~F~~	~~T~~
~~T~~	~~F~~	~~F~~
~~F~~	~~T~~	~~T~~
~~F~~	~~T~~	~~F~~
~~F~~	~~F~~	~~T~~
F	F	F

model

Proofs (symbolic manipulation of sentences) usually smaller than truth tables and thus often less work.

Satisfaction: Properties of Sentences

- Valid sentence
  - If and only if *every* interpretation satisfies it
- Contingent sentence
  - If and only if *some* interpretation satisfies it and *some* interpretation falsifies it
- Unsatisfiable sentence
  - If and only if *no* interpretation satisfies it

- Satisfiable
  - If and only if it is *either valid or contingent*
- Falsifiable
  - If and only if it is *either contingent or unsatisfiable*

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \vee q \vee \neg(p \wedge q)$
T	T	T
T	F	T
F	T	T
F	F	T

Satisfaction: Logical Equivalence

- Logical equivalence of sentences ϕ, ψ
  - If and only if every truth assignment that satisfies ϕ satisfies ψ *and* every truth assignment that satisfies ψ satisfies ϕ
  - Common equivalences:
    - Double negation: $p \Leftrightarrow \neg\neg p$
    - de Morgan's laws
      - $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$
      - $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
    - Implication: $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$
    - Equivalence: $(p \Leftrightarrow q) \Leftrightarrow \big((p \Rightarrow q) \wedge (q \Rightarrow p)\big)$
    - Distributive laws
      - $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
      - $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Satisfaction: Logical Entailment

- Logical entailment $\phi \models \psi$
  - Premise ϕ logically entails conclusion if and only if every interpretation that satisfies ϕ also satisfies ψ
    - Satisfying interpretations of ϕ need to be a subset of the satisfying interpretations of ψ
  - For sets: A set of premises entails a set of conclusions if and only if every interpretation that satisfies all premises also satisfies all conclusions
  - Examples
    - $(p \wedge q) \models (p \vee q)$
    - $p \models (p \vee q)$
    - $(p \wedge q) \models p$
    - $p \not\models (p \wedge q)$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Satisfaction: Consistency

- Consistency of sentences ϕ, ψ
  - If and only if there is a truth assignment that satisfies both ϕ and ψ
  - Examples
    - p is logically consistent with q
    - $(p \vee q)$ is logically consistent with $(\neg p \vee \neg q)$
    - $(p \Rightarrow q)$ is logically consistent with $(\neg p \vee q)$
    - p is not consistent with $\neg p$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Some Connections between These Concepts

- Equivalence Theorem
  - Sentences ϕ, ψ are logically equivalent if and only if $(\phi \Leftrightarrow \psi)$ is valid
- Unsatisfiability Theorem
  - $\Delta \models \psi$ if and only if $\Delta \cup \{\neg\psi\}$ is unsatisfiable
- Deduction Theorem
  - Sentence ϕ logically entails sentence ψ if and only if $(\phi \Rightarrow \psi)$ is valid
- Consistency Theorem
  - Sentence ϕ logically consistent with sentence ψ if and only if $(\phi \wedge \psi)$ is satisfiable

Normal Forms (NFs)

- *Literal*: either a proposition constant or its negation
  - $p, \neg p$
- Clause: either a literal or a disjunction of literals
  - $p, \neg p, \neg p \vee q$
- Negation normal form (NNF)
  - Sentences consisting of disjunctions and conjunctions of literals
    - p
    - $p \wedge q$
    - $(\neg p \wedge q) \vee \big(p \wedge (r \vee s)\big)$
- Conjunctive normal form (CNF)
  - Conjunction of clauses
    - p (single conjunct)
    - $(\neg p \vee q)$ (single conjunct)
    - $p \wedge q$
    - $(\neg p \vee q) \wedge (p \vee r) \wedge s$
- Disjunctive normal form (DNF)
  - Disjunction of conjunctions of literals
    - p (single disjunct)
    - $(p \wedge q)$ (single disjunct)
    - $\neg p \vee q$
    - $(\neg p \wedge q) \vee (p \wedge r) \vee s$

Normal Forms (NFs): Conversion into NNF

- Given a set of sentences
- For each sentence:
  1. Equivalences $\phi \Leftrightarrow \psi$ out
     - Replace with $(\phi \Rightarrow \psi) \wedge (q \Rightarrow \psi)$
  2. Implications $\phi \Rightarrow \psi$ out
     - Replace with $(\neg\phi \vee \psi)$
  3. Negations in
     - Replace $\neg\neg p$ with p
     - Apply de Morgan's laws
       - $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$
       - $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
- Result: sentences in NNF

- Example

$$\neg\big(p \wedge (q \Rightarrow r)\big)$$
$$\neg\big(p \wedge (\neg q \vee r)\big)$$
$$\neg p \vee \neg(\neg q \vee r)$$
$$\neg p \vee (\neg\neg q \wedge \neg r)$$
$$\neg p \vee (q \wedge \neg r)$$

  - In NNF

Normal Forms (NFs): Conversion into CNF / DNF

1. Convert into NNF
2. Distribute depending on NF
   - CNF: … disjunctions inward
     - Replace $\phi_1 \vee (\phi_2 \wedge \phi_3)$ with $(\phi_1 \vee \phi_2) \wedge (\phi_1 \vee \phi_3)$
   - DNF: … conjunctions inwards
     - Replace $\phi_1 \wedge (\phi_2 \vee \phi_3)$ with $(\phi_1 \wedge \phi_2) \vee (\phi_1 \wedge \phi_3)$

- Example

$$\neg\big(p \wedge (q \Rightarrow r)\big)$$
$$\neg\big(p \wedge (\neg q \vee r)\big)$$
$$\neg p \vee \neg(\neg q \vee r)$$
$$\neg p \vee (\neg\neg q \wedge \neg r)$$
$$\neg p \vee (q \wedge \neg r)$$

  - In NNF

What about CNF / DNF?

- Already in DNF
- To get a CNF, distribute $\neg p \vee$ inwards:

$$(\neg p \vee q) \wedge (\neg p \vee \neg r)$$

Interim Summary

- Syntax
  - Propositional vocabulary of proposition constants
  - Propositional sentence: combination of proposition constants, logical operators, parenthesis
  - Logical operators: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
- Semantics
  - Propositional interpretations: truth assignments to proposition constants
  - Sentential interpretations: evaluations of sentences given propositional interpretation
  - Satisfaction: all propositional interpretations that satisfy all sentences
    - Properties: valid, contingent, unsatisfiable; satisfiable, falsifiable
    - Comparison of sentences: equivalence, entailment, consistency; theorems for connections
- Normal forms
  - NNF, CNF, DNF

First-order Logic with Domain Constraints

Introducing quantifiers and domain constraints

* Based on “Lifted Inference and Learning in Statistical Relational Models” by Guy Van den Broeck, 2013

→ Some notions are used a bit differently than in some classical logic literature but I want to keep the slides consistent with this source

Propositional → First-order Logic (FOL)

- Motivation: Being able to talk about objects and relations among them
  - Example: in propositional logic
    - Premises
      - If Jack knows Jill, then Jill knows Jack.
      - Jack knows Jill.
    - Conclusion: Does Jill know Jack? (Yes.)
  - Example: in first-order (or relational) logic
    - Premises
      - If one person knows another person, then the other person knows the first person.
      - Jack knows Jill.
    - Conclusion: Does Jill know Jack? (Yes.)
- New linguistic features
  - (Logical) variables
    - X, Y
  - Quantifiers ∀, ∃
    - Followed by a logical variable
  - Example
    - Premises
      - $\forall X.\, \forall Y.\, \big(knows(X,Y) \Rightarrow knows(Y,X)\big)$
      - $knows(jack, jill)$
    - Conclusion: $knows(jill, jack)$?

FOL with Domain Constraints (FOL-DC)

- FOL has a domain of discourse that is possibly infinite
  - Quantifiers can range over infinite sets of objects
- Domain constraints
  - Restrict quantifiers to finite domains
- Why?
  - For probabilistic inference, infinity is hard to handle
    - Restriction allows for obtaining a well-defined probability distribution over the ground terms
    - Enables weighted model counting as a way to answer queries about probability (distributions)
  - Tasks like checking validity or satisfiability become decidable in FOL-DC
    - Undecidable in full FOL
    - Not the only decidable subclass that exists, but the one that we need for the lecture

Syntax: *Function-free* FOL

- Logical symbols
  - Logical connectives ¬, ∧, ∨, etc.
  - Quantifiers ∀, ∃
    - *Universal* quantifier ∀ (*for all*)
    - *Existence* quantifier ∃ (*there exists*)
    - Mostly, we will be dealing with ∀ later on
  - Truth values T and F
  - Set of logical variables $\boldsymbol{X} = \{X, Y, \ldots\}$
    - Usually from the end of the alphabet
  - (Parentheses and punctuation)
- Non-logical symbols
  - Set of predicate symbols p/n of *arity* n
    - Including propositional variables with arity $n = 0$
    - Sometimes called relation constants
  - A set of (proposition) constant symbols $\{a, b, c, \ldots\}$
    - Alphanumeric, usually starting with a letter from the beginning of the alphabet; numbers now also allowed
    - Sometimes called object constants
    - With a natural order on the constants

Syntax: *Function-free* FOL

- Signature: set of constants together with a set of predicates including arity
- Sentences no longer of one type
  - C.f., propositional logic
  - *Relational* sentences
    - n-ary predicate with n logical variables or constants in parentheses and with commas
  - *Logical* sentences
    - Relational sentences possibly combined with logical operators
  - *Quantified* sentences
    - Logical sentences that contain quantifiers for some or all logical variables occurring
- Examples
  - Signature
    - Constants: $jack, jill$
    - Predicate: $knows/2, raining/0$
  - Logical variables: X, Y
  - Relational sentence
    - $knows(X, Y)$
    - $knows(jack, X)$
  - Logical sentence
    - $knows(X, Y) \Rightarrow knows(Y, X)$
  - Quantified sentence
    - $\forall X. \forall Y. \big(knows(X, Y) \Rightarrow knows(Y, X)\big)$

Syntax: *Function-free* FOL-DC

- Function-free FOL extended with
  - Logical symbols
    - Less-than $<$
    - Set membership and inclusion $\in$, $\subseteq$
    - Set of domain variables $\mathbf{D} = \{\mathrm{D}, \mathrm{F}, \ldots\}$
    - Set operations $\cup$, $\cap$, $\setminus$
  - Non-logical symbols (signature)
    - Set of "sets of constants" symbols $\{\{a, b, c\}, \{a, d\}, \emptyset, \ldots\}$

- These constructs only occur when restricting domains using so-called constraint sets

Syntax: FOL-DC – Grammar

- Logical term
  - Logical variable or constant
  - Refers to objects in the domain of discourse (scenario)
  - Examples
    - Logical variables: X, Y
    - Constants: $jack, jill$
- Logical atom (*relational sentence*)
  - $p(t_1, \ldots, t_n)$
  - Apply predicate p/n to n-tuple of logical term arguments t_i
  - Examples
    - Predicate with logical variables only as arguments: $knows(X, Y)$
    - Predicate with a logical variable and a constant as arguments: $knows(jack, X)$
    - Predicate with constants only as arguments: $knows(jack, jill)$
    - 0-ary predicate (propositional atom): $raining$

Syntax: FOL-DC – Grammar

- Domain term
  - Domain variable, set of constants, or combination of two other domain terms t_d, t'_d in the form of
    $$t_d \cup t'_d, t_d \cap t'_d, t_d \setminus t'_d$$
  - Refers to sets of objects in the domain of discourse
  - Examples
    - Domain variables: D, E
    - Sets of constants: $\{a, b, c\}$, $\{jill, jack\}$
    - Combination of domain terms: $\mathrm{D} \cup \mathrm{E}$, $\mathrm{D} \cup \{jill, jack\}$

- Domain atom
  - Between logical terms t_l, t'_l
    $$t_l = t'_l$$
    $$t_l < t'_l$$
    - $X = jill$
    - $X < Y$ (natural order on constants needed)
  - Between domains terms t_d, t'_d
    $$t_d = t'_d$$
    $$t_d \subseteq t'_d$$
    - $\mathrm{D} = \mathrm{E}, \mathrm{E} \subseteq \mathrm{D} \cup \{jill, jack\}$
  - Between logical terms t_l, domain term t_d
    $$t_l \in t_d$$
    - $X \in \{jill, jack\}$

Syntax: FOL-DC – Grammar

- Domain constraint
  - Domain atom or its negation ($\neq, \notin, \not\subseteq$)
- Constraint set
  - Conjunction of domain constraints
    - (restricted to conjunction for the sake of simplicity)
    - (more expressive constraint languages would have the purpose to express certain constraint sets much more precisely)
- Example
  - $X \in \mathrm{Bird} \wedge X \neq kiwi$

Syntax: FOL-DC – Grammar

- (Well-formed) formula
  - Logical atom
  - If φ, ψ formulas, then the following are
    - $\neg\varphi$
    - *Extensional conjunction* $\varphi \wedge \psi$ or
    - *Extensional disjunction* $\varphi \vee \psi$
  - If φ formula, $\boldsymbol{V}$ a (possibly empty) set of (logical or domain) variables, and cs a constraint set that contains at least one domain atom of the form $V = t$, $V \in t$, or $V \subseteq t$ for every $V \in \boldsymbol{V}$, the following are
    - *Intensional conjunction* $\forall \boldsymbol{V}, cs : \varphi$
    - *Intensional disjunction* $\exists \boldsymbol{V}, cs : \varphi$
- A variable is bound
  - If it is quantified by an enclosing intensional conjunction or disjunction
- A variable is free
  - If it is not bound
- Sentence
  - Formula without free variables
- Formula is ground
  - If it does not contain any variables

Syntax: FOL-DC – Grammar

- Examples:
  - Domain atom:
    - $\mathrm{D} = \{jack, jill\}$
  - Logical atom:
    - $knows(X, Y), knows(jack, jill)$
  - *Extensional disjunction:*
    - $raining \vee snowing$
  - *Intensional conjunction:*

    $\forall X, Y, X \in \mathrm{D} \wedge Y \in \mathrm{D}.$

    $\big(knows(X, Y) \Rightarrow knows(Y, X)\big)$
    - Logical variables: X, Y
    - Constraint set: $X \in \mathrm{D} \wedge Y \in \mathrm{D}$

- Consider formula

  $\forall X, X \in \mathrm{D}.$

  $\big(knows(X, Y) \Rightarrow knows(Y, X)\big)$
  - Bound variable: X
  - Free variable: Y
    - Essentially universally quantified
  - Not a sentence because Y is free
    - Sentence:

      $\forall X, Y, X \in \mathrm{D} \wedge Y \in \mathrm{D}.$

      $\big(knows(X, Y) \Rightarrow knows(Y, X)\big)$
  - Ground:
    - $knows(jack, jill) \Rightarrow knows(jill, jack)$
    - $raining \vee snowing$

Some More Information about Quantifiers

- Quantifiers can be nested
  - $\forall X.\left(\exists Y.knows(X,Y)\right)$
- Precedence: quantifiers have higher precedence than logical operators
- Quantifier reversal
  - $\forall X.\forall Y.p(X,Y) \Leftrightarrow \forall Y.\forall X.p(X,Y)$
  - $\exists X.\exists Y.p(X,Y) \Leftrightarrow \exists Y.\exists X.p(X,Y)$
- Existential distribution
  - $\exists Y.\forall X.p(X,Y) \Rightarrow \forall X.\exists Y.p(X,Y)$
- Negation distribution
  - $\neg\forall X.\phi \Leftrightarrow \exists X.\neg\phi$
  - $\neg\exists X.\phi \Leftrightarrow \forall X.\neg\phi$

Further Terminology

- Literal l
  - Logical atom a or its negation $\neg a$
- Clause
  - Disjunction of literals
- Theory in conjunctive normal form (CNF)
  - Conjunction of clauses
- Term
  - Conjunction of literals
- Theory in disjunctive normal form (DNF)
  - Disjunction of terms
- Theory or knowledge base
  - Conjunction of formulas

- Examples
  - Literals
    - $knows(X,Y), \neg knows(X,Y), raining$
  - Clause
    - $raining \vee knows(jill,Y)$
  - Term
    - $knows(X,Y) \wedge knows(Y,X)$
  - Knowledge base

$$knows(jack, jill)$$
$$\wedge$$
$$\forall X, Y, X \in \mathrm{D} \wedge Y \in \mathrm{D}.$$
$$\big(knows(X,Y) \Rightarrow knows(Y,X)\big)$$

Semantics: Herbrand Base

- Herbrand base
  - Set of all grounded logical atoms that can be formed
  - Leads to a propositional language
- Instance
  - Consistently replace every occurrence of a free variable with a constant
    - *Consistent replacement*: If one occurrence of a variable replaced by a constant, then all occurrences of that variable replaced by the same constant

- Example
  - Herbrand base
    - Constants: $jack, jill$
    - Predicate: $knows/2$, $raining/0$
    - $knows(jack, jill), knows(jack, jack), knows(jill, jack). knows(jill, jill), raining$
  - Example instances
    - $knows(X, Y)$
      - $knows(jack, jill)$
      - $knows(jill, jill)$
    - $p(X) \Rightarrow q(X)$
      - $p(a) \Rightarrow q(a)$

Semantics: Truth Assignment

- Truth assignments to a Herbrand base works as in propositional languages
  - Propositional truth assignments
  - Sentential truth assignments
    - Same operator semantics
  - Universally quantified formula true for a truth assignment if and only if every instance of the scope of the quantified formula is true for that assignment
  - Existentially quantified formula is true for a truth assignment if and only if some instance of the scope of the quantified sentence is true for that assignment

- Example
  - Formula

$$\forall X, Y, X \in \mathrm{D} \wedge Y \in \mathrm{D}.$$
$$\big(knows(X,Y) \Rightarrow knows(Y,X)\big)$$

  - $knows(jack, jill), knows(jack, jack),$ $knows(jill, jill), knows(jill, jack)$
  - Truth assignment TTTT
    - Formula true
  - Truth assignment FTTF
    - Formula true
  - Truth assignment FTTT
    - Formula false

Evaluation and satisfaction then work as before. Thus, properties may also hold or not and formulas may be compared (equivalence, entailment, consistency).

Interim Summary

- Function-free first-order logic with domain constraints contains new constructs
  - Logical variables
  - Quantifiers
  - Domain variables and domain constraints
  - Quantifier equivalences
- Semantics
  - Herbrand base: grounding
  - Interpretations work as before
    - Quantifiers semantics
    - Evaluation and satisfaction work as before
      - Properties, comparison as well

Overview: 2. Foundations

A. *Logic*
   - Propositional logic: alphabet, grammar, normal forms, rules
   - First-order logic: introducing quantifiers, domain constraints

B. *Probability theory*
   - Modelling: (conditional) probability distributions, random variables, marginal and joint distributions
   - Inference: axioms and basic rules, Bayes theorem, independence

C. *Probabilistic graphical models*
   - Syntax, semantics
   - Inference problems