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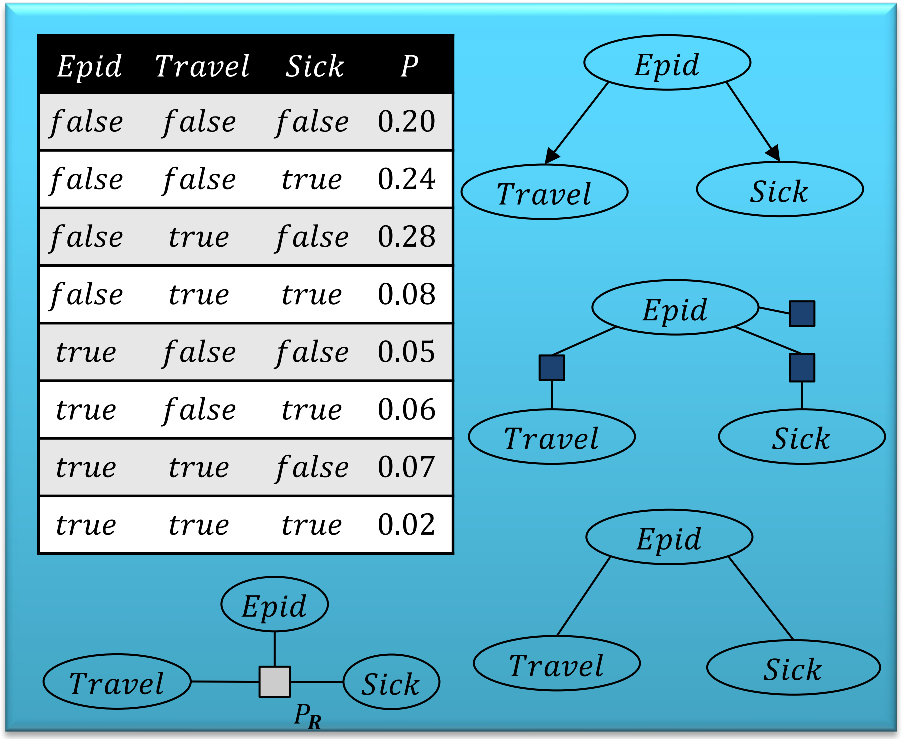
Dynamic Probabilistic Relational Models



Epid	Travel	Sick	P
false	false	false	0.20
false	false	true	0.24
false	true	false	0.28
false	true	true	0.08
true	false	false	0.05
true	false	true	0.06
true	true	false	0.07
true	true	true	0.02

Foundations: Probabilistic Graphical Models

Marcel Gehrke

Contents

Overview: 2. Foundations

A. *Logic*
   - Propositional logic: alphabet, grammar, normal forms, rules
   - First-order logic: introducing quantifiers, domain constraints

B. *Probability theory*
   - Modelling: (conditional) probability distributions, random variables, marginal and joint distributions
   - Inference: axioms and basic rules, Bayes theorem, independence

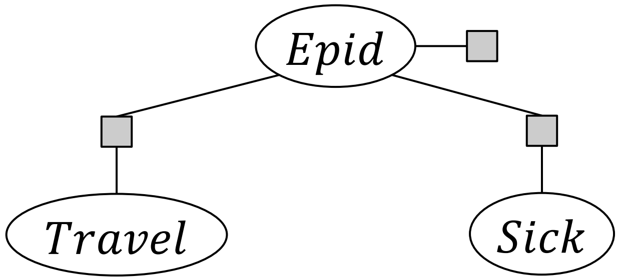
C. ***Probabilistic graphical models***
   - Syntax, semantics
   - Inference problems

Propositional Graphical Models (PGMs)

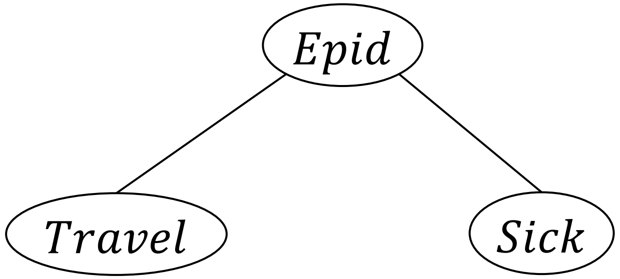
- *Factorisation* of a full joint according to (conditional) independences in the full joint

$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

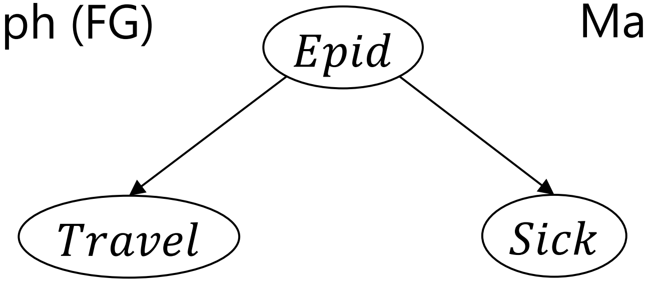
Full joint probability distribution



Factor graph (FG)



Markov network (MN)

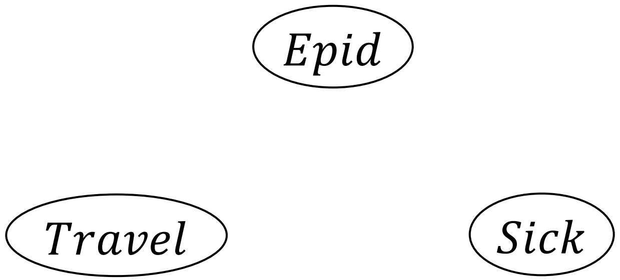


Bayesian network (BN)

Variants of PGMs

Random Variables

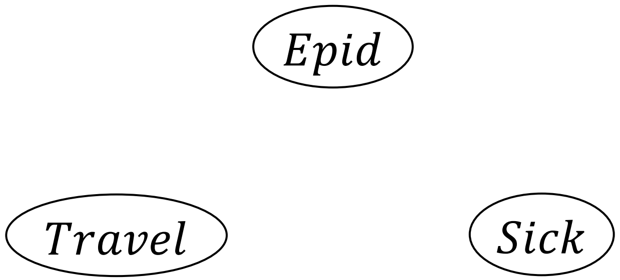
- Characterise scenario by set of random variables
  - $\boldsymbol{R} = \{R_1, \dots, R_N\}$
  - Often depicted as ellipses
  - E.g., $\{Epid, Travel.eve, Sick.eve\}$
- Possible values a random variable can take = range (or valuation)
  - $ran(R) = Val(R) = \{v_1, \dots, v_m\}$
  - If $|ran(R)| = 2$, often called Boolean range
  - E.g., $ran(Epid) = ran(Travel.eve) = ran(Sick.eve) = \{true, false\}$



Events

- Observing or setting a random variable to a specific range value = event
  - $R = r, r \in \mathcal{R}(R)$
  - Shorthand:
    - If R clear from context, we write r instead of $R = r$
      - If $ran(R)$ Boolean, we write r for $R = true$ and $\neg r$ for $R = false$
  - E.g.,

$$\begin{array}{c|ccc} Epid = e & Epid = true & & Epid = false \\ e & epid & & \neg epid \end{array}$$



- Setting range values for a set of random variables, one value for each variable = compound event

Full Joint Probability Distribution

- 1 world = compound event for $\boldsymbol{R}$

$$\begin{matrix} epid \\ \neg travel \\ \neg\, sick \end{matrix}$$

- Specify a probability for a world

$$P(epid, \neg travel, \neg sick) = 0.05$$

– Joint probability distribution $P_{\boldsymbol{R}}$ over all (l) possible worlds
  - $\sum_{i=1}^{l} P(w_i) = 1$
    - w_i: compound event for $\boldsymbol{R}$

How large is l?

$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
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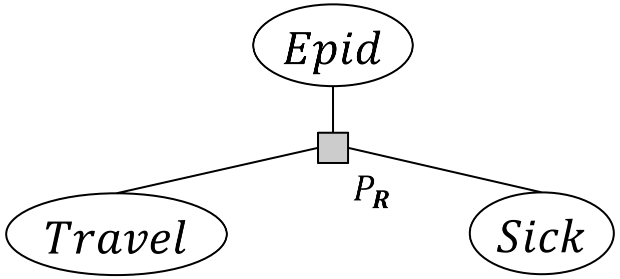
$true$	$true$	$true$	0.02



Space Complexity

- Joint probability distribution $P_{\boldsymbol{R}}$ over all (l) possible worlds
  - $\sum_{i=1}^{l} P(w_i) = 1$
  - w_i: compound event for $R \in \boldsymbol{R}$
- Space complexity: $O(r^N)$
  - $r = \max_{R \in \boldsymbol{R}} |ran(R)|$
  - $N = |\boldsymbol{R}|$
  - Derivation:

$$\underbrace{\prod_{R \in \boldsymbol{R}} |ran(R)|}_{\text{Exact size}} \leq \prod_{R \in \boldsymbol{R}} \max_{R \in \boldsymbol{R}} |ran(R)| = \prod_{R \in \boldsymbol{R}} r = r^{|\boldsymbol{R}|} = r^N$$

Exponential in N!

$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
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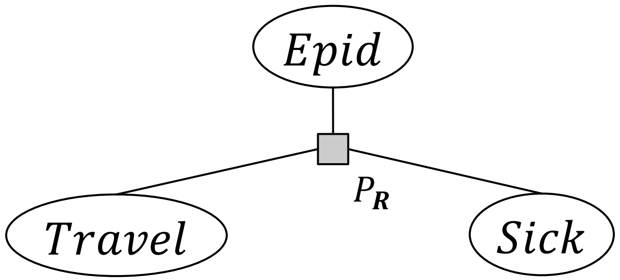

Inference Tasks

- Query Answering Problem
  - Compute an answer to a query given full joint probability distribution $P_{\boldsymbol{R}}$
    - Query for a marginal (conditional) probability (distribution)
      - Marginal probability of events: $P(epid)$
      - Marginal probability distribution of random variables: $P(Epid, Travel)$
      - Marginal conditional probability of events given random variables or events: $P(epid|sick)$
      - Marginal conditional probability distribution of random variables given random variables or events: $P(Sick|Epid)$
- Next slides
  - Syntax of queries
  - Solving an instance of a query answering problem
    - Preview: Eliminate all non-query terms

$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
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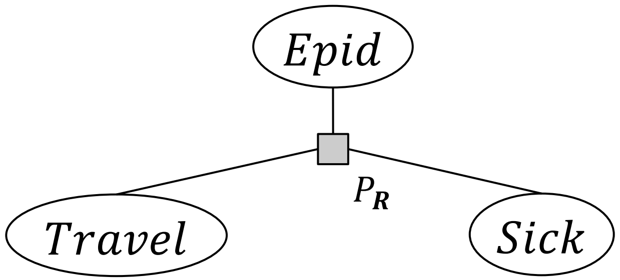
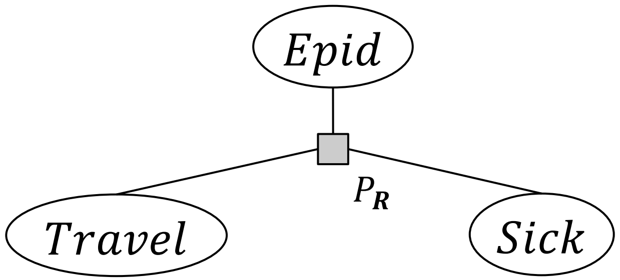


Query Syntax

- Marginal probability (distribution) w.r.t. $P_{\boldsymbol{R}}$: $P(\boldsymbol{S})$
  - $rv(\boldsymbol{S}) \subseteq \boldsymbol{R}$
    - $rv(.)$: shorthand notation to refer to random variables in the input
  - $\boldsymbol{S}$: random variables or events
  - Example: $P(Epid, Travel)$
- Conditional marginal probability distribution w.r.t. $P_{\boldsymbol{R}}$: $P(\boldsymbol{S}|\boldsymbol{T})$
    - $rv(\boldsymbol{S}, \boldsymbol{T}) \subseteq \boldsymbol{R}$
    - $\boldsymbol{S} \cap \boldsymbol{T} = \emptyset$
  - $\boldsymbol{S}$: random variables or events
  - $\boldsymbol{T}$: random variables or events $\boldsymbol{t}$ (considered observations, called evidence)
  - Example: $P(Travel|Epid)$, $P(Epid|sick)$

$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02



Answering Marginal Queries

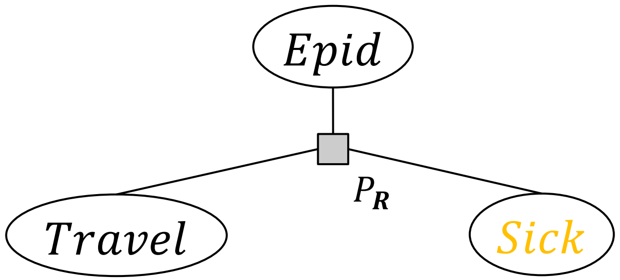
- Solving $P(\boldsymbol{S})$: Eliminate all non-query terms $\boldsymbol{U}$
  - $\boldsymbol{U} = \boldsymbol{R} \setminus rv(\boldsymbol{S})$

$$P(\boldsymbol{S}) = \sum_{\boldsymbol{u} \in ran(\boldsymbol{U})} P_{\boldsymbol{R}}(\boldsymbol{S}, \boldsymbol{U} = \boldsymbol{u})$$

  - E.g., query $P(Epid, Travel) \rightarrow \boldsymbol{U} = \{Sick\}$

$$P(Epid, Travel) = \sum_{v \in ran(Sick)} P(Epid, Travel, Sick = v)$$

$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

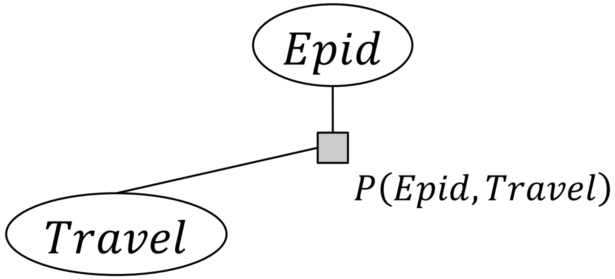


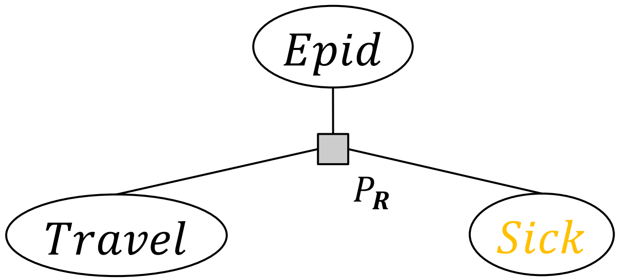
Answering Marginal Queries

$$P(Epid, Travel) = \sum_{v \in ran(Sick)} P(Epid, Travel, Sick = v)$$

Epid	Travel	Sick	P
false	false	false	0.20
false	false	true	0.24
false	true	false	0.28
false	true	true	0.08
true	false	false	0.05
true	false	true	0.06
true	true	false	0.07
true	true	true	0.02

Epid	Travel	P
false	false	0.44
false	true	0.36
true	false	0.11
true	true	0.09





Answering Marginal Queries

- If $\boldsymbol{S}$ in $P(\boldsymbol{S})$ consists of events, consider only the worlds that are consistent with the events
  - E.g., query $P(epid)$ with $\boldsymbol{U} = \{Travel, Sick\}$

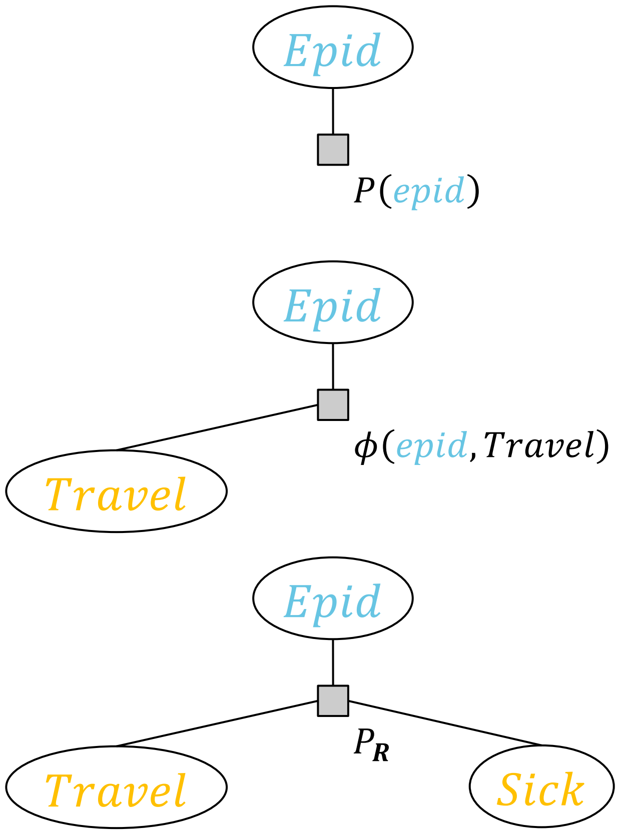
$$P(epid) = \sum_{v_t \in ran(Travel)} \sum_{v_s \in ran(Sick)} P(epid, Travel = v_t, Sick = v_s)$$

$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

No probability distribution at this point (in the middle of the computation)

$Epid$	$Travel$	ϕ
$true$	$false$	0.11
$true$	$true$	0.09

$Epid$	P
$true$	0.20



Answering Conditional Queries

- Solving $P(\boldsymbol{S}|\boldsymbol{T})$ w.r.t. $P_{\boldsymbol{R}}$:
  - $P(\boldsymbol{S}|\boldsymbol{T}) = \frac{P(\boldsymbol{S},\boldsymbol{T})}{P(\boldsymbol{T})}$
  - $P(\boldsymbol{T})$ normalising constant
- Reduces to computing two marginal queries: $P(\boldsymbol{S},\boldsymbol{T})$, $P(\boldsymbol{T})$
- Eliminate all non-query terms $\boldsymbol{U}$ and normalise
  - $\boldsymbol{U} = \boldsymbol{R} \setminus rv(\boldsymbol{S},\boldsymbol{T})$

$$P(\boldsymbol{S}|\boldsymbol{T}) = \frac{1}{P(\boldsymbol{T})} \sum_{\boldsymbol{u} \in ran(\boldsymbol{U})} P_{\boldsymbol{R}}(\boldsymbol{S},\boldsymbol{T},\boldsymbol{U}=\boldsymbol{u})$$

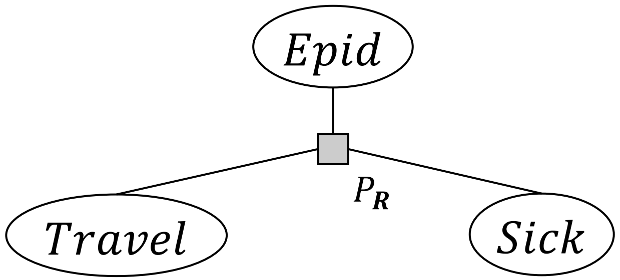
  - If $\boldsymbol{T} = \boldsymbol{t}$ (evidence), drop the rows where $\boldsymbol{T} \neq \boldsymbol{t}$ and columns $\boldsymbol{T}$
    - Called evidence *absorption*, reduces dimension of $P_{\boldsymbol{R}}$ as $\boldsymbol{T}$ disappears
    - Equal to setting probabilities to 0 where $\boldsymbol{T} \neq \boldsymbol{t}$ and summing out $\boldsymbol{T}$

- As $\boldsymbol{T}$ often $\boldsymbol{t}$, $P(\boldsymbol{T}) = P(\boldsymbol{t})$ is constant for $P(\boldsymbol{S}|\boldsymbol{t})$, thus called *normalising constant*.
- As $P(\boldsymbol{t})$ is a sum expression in itself, it is often abbreviated:

$P(\boldsymbol{S}|\boldsymbol{T}) = \frac{1}{Z}\Sigma \ldots$ (partition function)

$P(\boldsymbol{S}|\boldsymbol{T}) = \alpha\Sigma \ldots$

$P(\boldsymbol{S}|\boldsymbol{T}) \propto \Sigma \ldots$ (proportional to)



Answering Conditional Queries: Normalisation

- Normalise by $P(\boldsymbol{T})$: for each $\boldsymbol{t} \in ran(\boldsymbol{T})$, for each $\boldsymbol{s} \in ran(\boldsymbol{S})$, compute $\frac{P(\boldsymbol{s},\boldsymbol{t})}{P(\boldsymbol{t})}$

$$P(Travel|Epid) = \frac{1}{P(Epid)} \underbrace{\sum_{v_s \in ran(Sick)} P(Epid, Travel, Sick = v_s)}_{\phi}$$

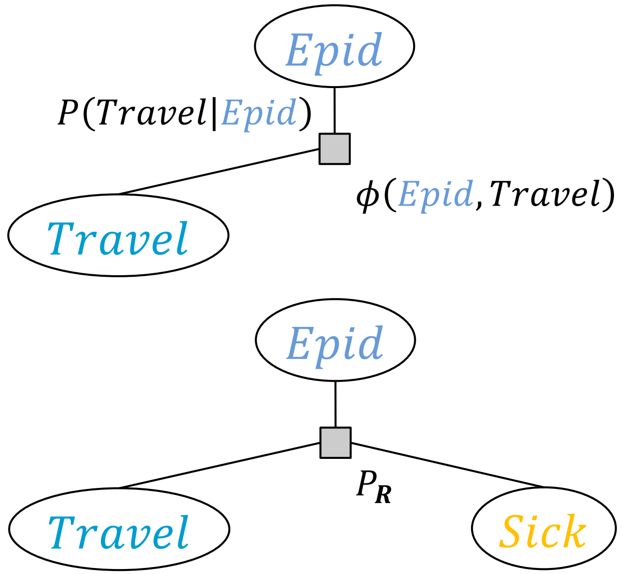
  - for each $e \in ran(Epid)$: for each $t \in ran(Travel)$: divide $\phi(e,t)$ by $P(e) = \phi(e,t) + \phi(e,\neg t)$

What do we need to do?

$Epid$	$Travel$	ϕ
$false$	$false$	0.44
$false$	$true$	0.36
$true$	$false$	0.11
$true$	$true$	0.09

$P(Travel\|Epid)$
$\frac{0.44}{0.44+0.36} = 0.55$
$\frac{0.36}{0.44+0.36} = 0.45$
$\frac{0.11}{0.11+0.09} = 0.55$
$\frac{0.09}{0.11+0.09} = 0.45$

$Epid$	$P(travel\|Epid)$	$P(\neg travel\|Epid)$
$false$	0.45	$1 - 0.45 = 0.55$
$true$	0.45	$1 - 0.45 = 0.55$



Answering Conditional Queries: Evidence – Absorption

$$P(Epid|sick) \propto \sum_{v \in ran(Travel)} P(Epid, Travel = v, sick)$$

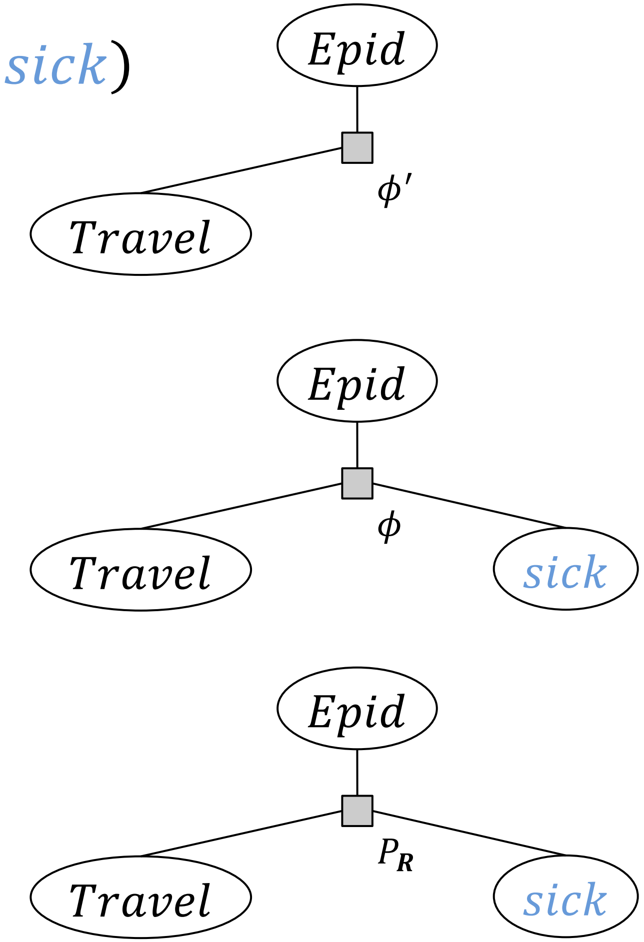
$Epid$	$Travel$	$Sick$	P
~~$false$~~	~~$false$~~	~~$false$~~	~~0.20~~
$false$	$false$	$true$	0.24
~~$false$~~	~~$true$~~	~~$false$~~	~~0.28~~
$false$	$true$	$true$	0.08
~~$true$~~	~~$false$~~	~~$false$~~	~~0.05~~
$true$	$false$	$true$	0.06
~~$true$~~	~~$true$~~	~~$false$~~	~~0.07~~
$true$	$true$	$true$	0.02

$Epid$	$Travel$	ϕ'
$false$	$false$	0.24
$false$	$true$	0.08
$true$	$false$	0.06
$true$	$true$	0.02

select + project

ϕ	Σ_s
0	$0 + 0.24$
0.24	
0	$0 + 0.08$
0.08	
0	$0 + 0.06$
0.06	
0	$0 + 0.02$
0.02	

set to 0 + sum out



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Answering Conditional Queries: Evidence – Normalisation

$$P(Epid|sick) \propto \sum_{v \in ran(Travel)} P(Epid, Travel = v, sick)$$

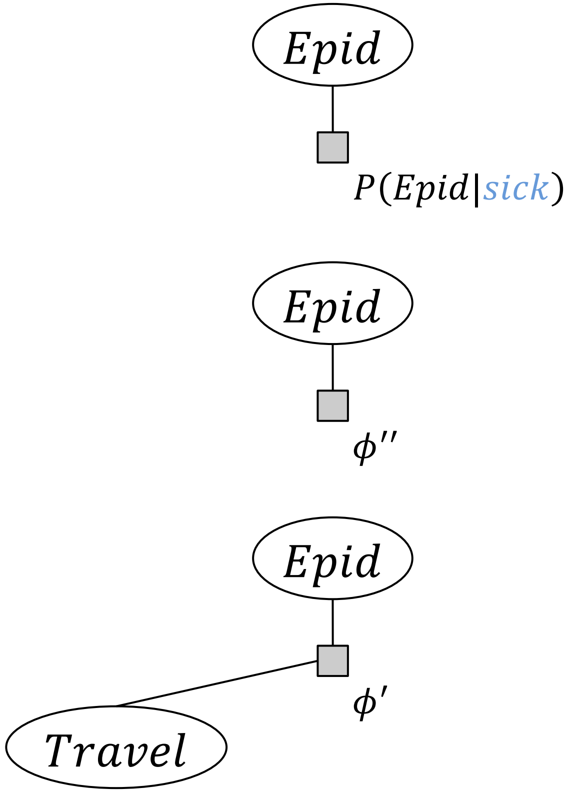
normalise

$Epid$	$Travel$	ϕ'
$false$	$false$	0.24
$false$	$true$	0.08
$true$	$false$	0.06
$true$	$true$	0.02

+

$Epid$	ϕ''
$false$	0.32
$true$	0.08

$Epid$	P
$false$	$\frac{0.32}{0.32 + 0.08} = 0.8$
$true$	$\frac{0.08}{0.32 + 0.08} = 0.2$



for each $t \in ran(T)$, for each $s \in ran(S)$, compute $\frac{P(s,t)}{P(t)}$

with $T = t$, only one case t

Runtime Complexity

$$P(\boldsymbol{S}|\boldsymbol{T}) = \frac{1}{P(\boldsymbol{T})} \sum_{\boldsymbol{u} \in ran(\boldsymbol{U})} P_{\boldsymbol{R}}(\boldsymbol{S}, \boldsymbol{T}, \boldsymbol{U} = \boldsymbol{u})$$

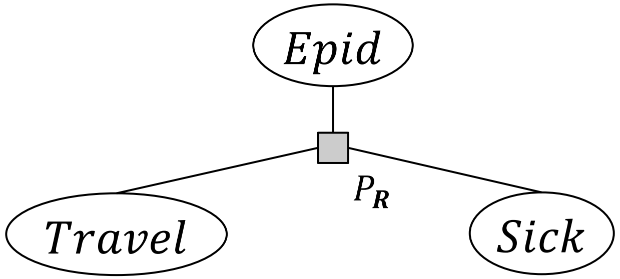
$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02

- Runtime complexity: $O(r^N)$
  - $r = \max_{R \in \boldsymbol{R}} |\mathcal{R}(R)|$
  - $N = |\boldsymbol{R}|$
  - Have to go through whole table; derivation as before

$$\prod_{R \in \boldsymbol{R}} |ran(R)| \leq \prod_{R \in \boldsymbol{R}} \max_{R \in \boldsymbol{R}} |ran(R)| = \prod_{R \in \boldsymbol{R}} r = r^{|\boldsymbol{R}|} = r^N$$

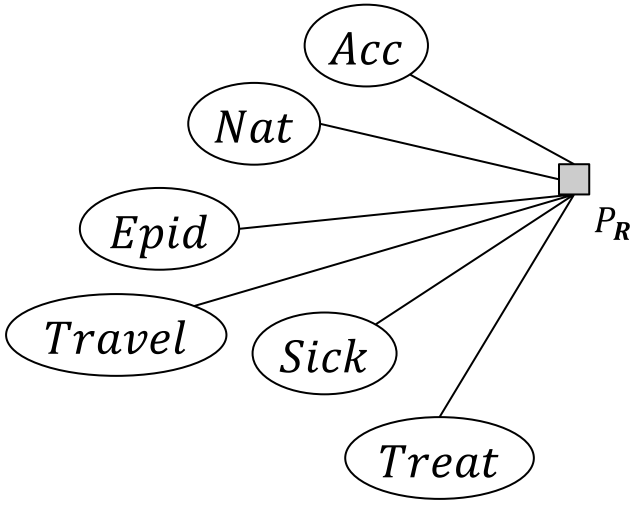
= Space complexity

Exponential in N!



Exponential Blowup!

Acc	Nat	$Treat$	$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	$false$	$false$	$false$	0.025
$false$	$false$	$false$	$false$	$false$	$true$	0.009
$false$	$false$	$false$	$false$	$true$	$false$	[illegible]
$false$	$false$	$false$	$false$	$true$	[illegible]	
$false$	$false$	$false$	$true$	[illegible]		
$false$	$false$	$false$	[illegible]			
$false$	$false$	[illegible]				
$false$	[illegible]					

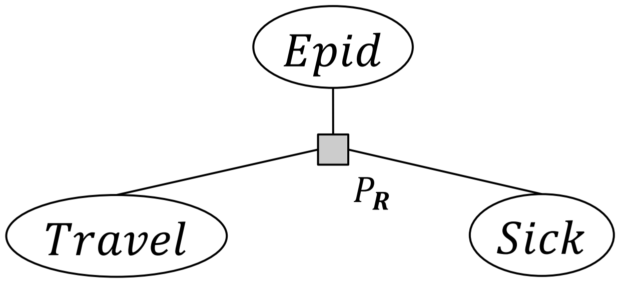


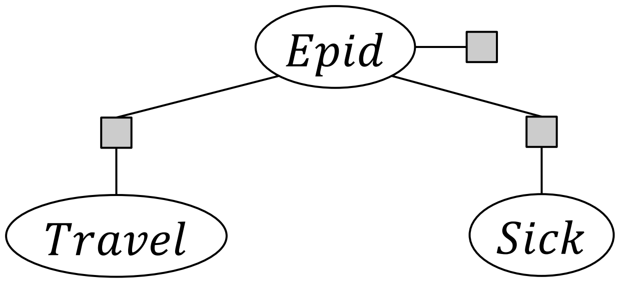
$2^6 = 64$ possible worlds

Adding relations means adding $Sick$, $Treat$, $Travel$ variables for each person, blowing up the model further

Compact Encoding

- Full joint probability distribution:
  Every random variable is connected with every other random variable!
- Factorise full joint probability distribution $P_{\boldsymbol{R}}$ using (conditional) independences
  - Independence: $P(\boldsymbol{R}_1, \boldsymbol{R}_2) = P(\boldsymbol{R}_1) \cdot P(\boldsymbol{R}_2)$ → denoted $(\boldsymbol{R}_1 \perp \boldsymbol{R}_2)$
  - Conditional independence: $P(\boldsymbol{R}_1, \boldsymbol{R}_2 | \boldsymbol{R}_3) = P(\boldsymbol{R}_1 | \boldsymbol{R}_3) \cdot P(\boldsymbol{R}_2 | \boldsymbol{R}_3)$ → denoted $(\boldsymbol{R}_1 \perp \boldsymbol{R}_2 | \boldsymbol{R}_3)$
  - Hidden in $P_{\boldsymbol{R}}$
    - Explicitly represent through factors and in graph
    - Full joint is then given by *product* of the factors





Excursion: Multiplication

- Join over arguments + product of probabilities:

$$\phi(R_1, \dots, R_l) = \phi_1(R_{11}, \dots, R_{1k}) \cdot \phi_2(R_{21}, \dots, R_{2m})$$

  - $\{R_1, \dots, R_l\} = \{R_{11}, \dots, R_{1k}\} \cup \{R_{21}, \dots, R_{2m}\}$
  - No common arguments = cross product of ranges
  - E.g.,

$$P(Epid, Travel) \cdot P(Travel, Sick)$$

Epid	*Travel*	*P*
false	*false*	0.10
false	*true*	0.20
true	*false*	0.30
true	*true*	0.40

•

Travel	*Sick*	*P*
false	*false*	0.25
false	*true*	0.30
true	*false*	0.35
true	*true*	0.10

Epid	*Travel*	*Sick*	*P*
false	*false*	*false*	$0.10 \cdot 0.25$

Excursion: Multiplication

- Join over arguments + product of probabilities:

$$\phi(R_1, \dots, R_l) = \phi_1(R_{11}, \dots, R_{1k}) \cdot \phi_2(R_{21}, \dots, R_{2m})$$

  - $\{R_1, \dots, R_l\} = \{R_{11}, \dots, R_{1k}\} \cup \{R_{21}, \dots, R_{2m}\}$
  - No common arguments = cross product of ranges
  - E.g.,

$$P(Epid, Travel) \cdot P(Travel, Sick)$$

$Epid$	$Travel$	P
$false$	$false$	0.10
$false$	$true$	0.20
$true$	$false$	0.30
$true$	$true$	0.40

•

$Travel$	$Sick$	P
$false$	$false$	0.25
$false$	$true$	0.30
$true$	$false$	0.35
$true$	$true$	0.10

$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	$0.10 \cdot 0.25$
$false$	$false$	$true$	$0.10 \cdot 0.30$

Excursion: Multiplication

- Join over arguments + product of probabilities:

$$\phi(R_1, \dots, R_l) = \phi_1(R_{11}, \dots, R_{1k}) \cdot \phi_2(R_{21}, \dots, R_{2m})$$

  - $\{R_1, \dots, R_l\} = \{R_{11}, \dots, R_{1k}\} \cup \{R_{21}, \dots, R_{2m}\}$
  - No common arguments = cross product of ranges
  - E.g., $P(Epid, Travel) \cdot P(Travel, Sick)$

$Epid$	$Travel$	P
$false$	$false$	0.10
$false$	$true$	0.20
$true$	$false$	0.30
$true$	$true$	0.40

•

$Travel$	$Sick$	P
$false$	$false$	0.25
$false$	$true$	0.30
$true$	$false$	0.35
$true$	$true$	0.10

$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	$0.10 \cdot 0.25$
$false$	$false$	$true$	$0.10 \cdot 0.30$
$false$	$true$	$false$	$0.20 \cdot 0.35$

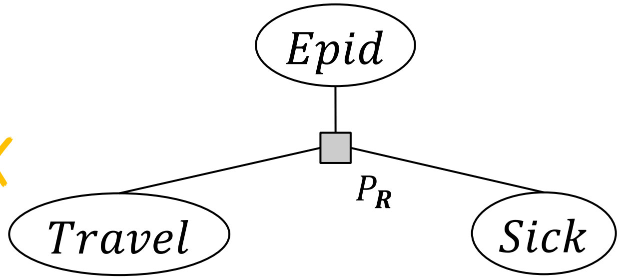
⋮

Independences: Examples

- Independence: $(Travel \perp Sick)$?
  - $P(Travel, Sick) \stackrel{?}{=} P(Travel) \cdot P(Sick)$
    $\rightarrow$ only $true, true$ case:
    - $P(travel, sick) = 0.08 + 0.02 = 0.1$
    - $P(travel) = 0.28 + 0.08 + 0.07 + 0.02 = 0.45$
    - $P(sick) = 0.24 + 0.08 + 0.06 + 0.02 = 0.4$
    - $P(travel) \cdot P(sick) = 0.45 \cdot 0.4 = 0.18 \neq 0.1$ ✗

- Conditional independence $(Travel \perp Sick | Epid)$?
  - $P(Travel, Sick | Epid) \stackrel{?}{=} P(Travel | Epid) \cdot P(Sick | Epid)$ ✗

$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.20
$false$	$false$	$true$	0.24
$false$	$true$	$false$	0.28
$false$	$true$	$true$	0.08
$true$	$false$	$false$	0.05
$true$	$false$	$true$	0.06
$true$	$true$	$false$	0.07
$true$	$true$	$true$	0.02



Independences: Examples

- $Travel \perp Sick \mid Epid$?
  - $P(Travel, Sick | Epid) \stackrel{?}{=} P(Travel | Epid) \cdot P(Sick | Epid)$ ✓

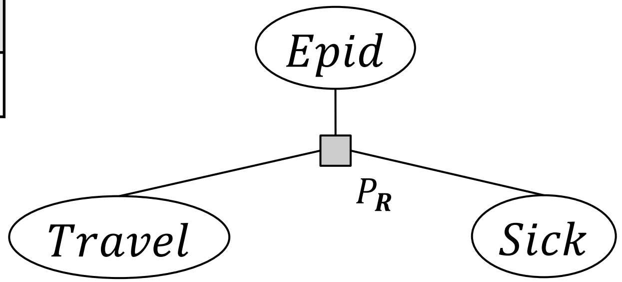
$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.6375
$false$	$false$	$true$	0.1125
$false$	$true$	$false$	0.2125
$false$	$true$	$true$	0.0375
$true$	$false$	$false$	0.45
$true$	$false$	$true$	0.15
$true$	$true$	$false$	0.3
$true$	$true$	$true$	0.1

$Epid$	$Travel$	P
$false$	$false$	0.75
$false$	$true$	0.25
$true$	$false$	0.6
$true$	$true$	0.4

$Epid$	$Sick$	P
$false$	$false$	0.85
$false$	$true$	0.15
$true$	$false$	0.75
$true$	$true$	0.25

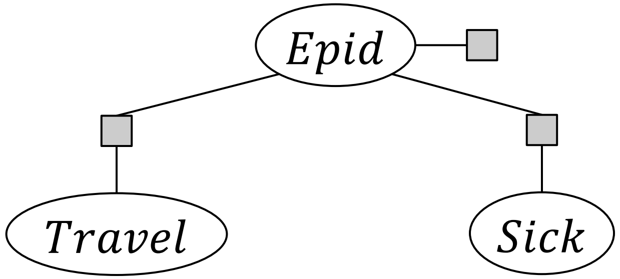
P
$0.75 \cdot 0.85 = 0.6375$
$0.75 \cdot 0.15 = 0.1125$
$0.25 \cdot 0.85 = 0.2125$
$0.25 \cdot 0.15 = 0.0375$
$0.6 \cdot 0.75 = 0.45$
$0.6 \cdot 0.25 = 0.15$
$0.4 \cdot 0.75 = 0.3$
$0.4 \cdot 0.25 = 0.1$

$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.51
$false$	$false$	$true$	0.09
$false$	$true$	$false$	0.17
$false$	$true$	$true$	0.03
$true$	$false$	$false$	0.09
$true$	$false$	$true$	0.03
$true$	$true$	$false$	0.06
$true$	$true$	$true$	0.02



Independences: Examples

- Factorise the full joint into its factors based on independences:
  - $P(Epid, Travel, Sick) = P(Epid) \cdot P(Travel|Epid) \cdot P(Sick|Epid)$



$Epid$	P
$false$	0.8
$true$	0.2

$Epid$	$Travel$	P
$false$	$false$	0.75
$false$	$true$	0.25
$true$	$false$	0.6
$true$	$true$	0.4

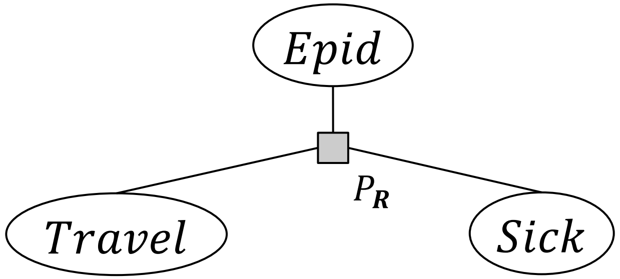
$Epid$	$Sick$	P
$false$	$false$	0.85
$false$	$true$	0.15
$true$	$false$	0.75
$true$	$true$	0.25

$P(epid)$
0.2

$Epid$	$P(travel\|Epid)$
$false$	0.25
$true$	0.4

$Epid$	$P(sick\|Epid)$
$false$	0.15
$true$	0.25

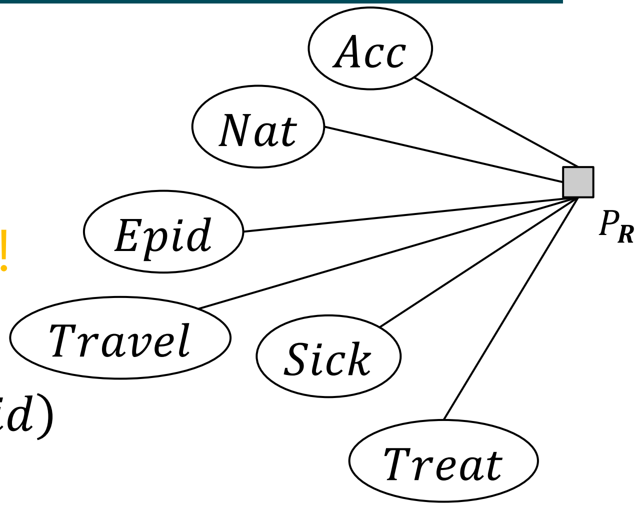
$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	0.51
$false$	$false$	$true$	0.09
$false$	$true$	$false$	0.17
$false$	$true$	$true$	0.03
$true$	$false$	$false$	0.09
$true$	$false$	$true$	0.03
$true$	$true$	$false$	0.06
$true$	$true$	$true$	0.02

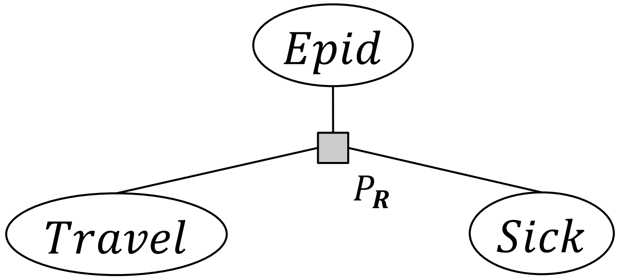


  - Usually fewer entries to store

Finding a Compact Encoding

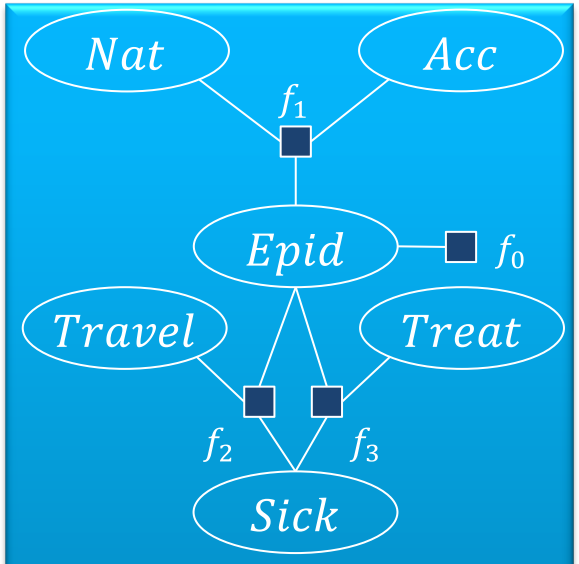
- At beginning: Everything connected with everything in full joint
- Find (conditional) independences in $P_{\boldsymbol{R}}$
  - Check every possible combination → Combinatorial explosion!
    - E.g., (many more)
      - $(Epid \perp Travel, Sick)$, $(Travel \perp Epid, Sick)$, $(Sick \perp Travel, Epid)$
  - Partitions $P_{\boldsymbol{R}}$ into a set of factors
    - Deletes connections between random variables
- Alternative:
  - Start with no connections, add factors
  - More later (→ *Section 8: Lifted Learning*)
- For now, assume that we have a factorised model





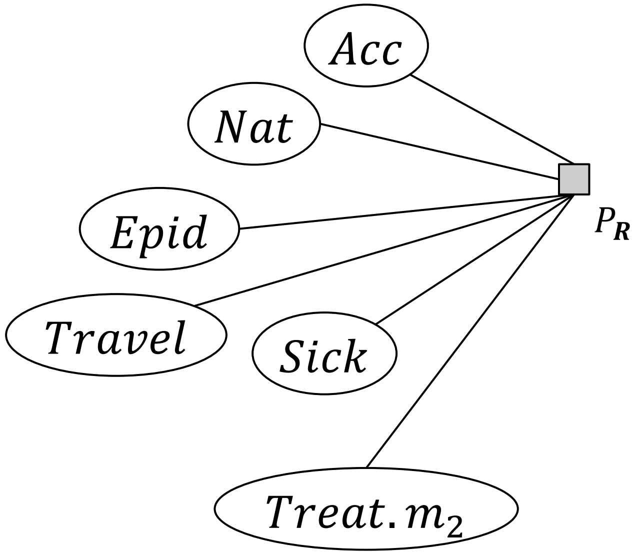
Exponential Blowup! → Sparse Encoding

Acc	Nat	$Treat$	$Epid$	$Travel$	$Sick$	P
$false$	$false$	$false$	$false$	$false$	$false$	0.025
$false$	$false$	$false$	$false$	$false$	$true$	0.009
$false$	$false$	$false$	$false$	$true$	$false$	[illegible]
$false$	$false$	$false$	$false$	$true$	[illegible]	
$false$	$false$	$false$	$true$	[illegible]		
$false$	$false$	$false$	[illegible]			
$false$	$false$	[illegible]				
$false$	[illegible]					



$3 \cdot 2^3 + 2$

$=$ **26 entries**



$2^6 = 64$ **possible worlds**

Adding relations means adding $Sick$, $Treat$, $Travel$ variables for each person, blowing up the model further

Model Representation: Factors

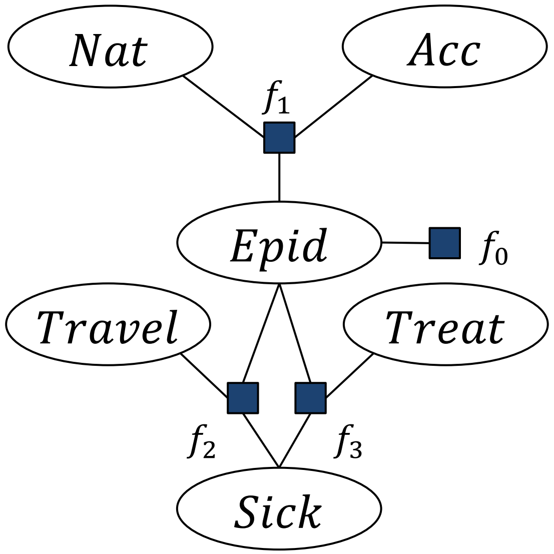
- Given set of random variables $\boldsymbol{R} = \{R_1, \ldots, R_N\}$

Syntax:

- Set of factors $F = \{f_i\}_{i=1}^{n'}$ = model
- Factor $f = \phi(R_1, \ldots, R_k)$
  - Arguments $R_1, \ldots, R_k \in \boldsymbol{R}$
  - Potential function

$$\phi: \times_{i=1}^{k} ran(R_i) \longrightarrow \mathbb{R}^{0,+}$$

    - At least one potential > 0
    - Write as table, list, …
    - Not required to be a probability distribution



Model Representation: Factors

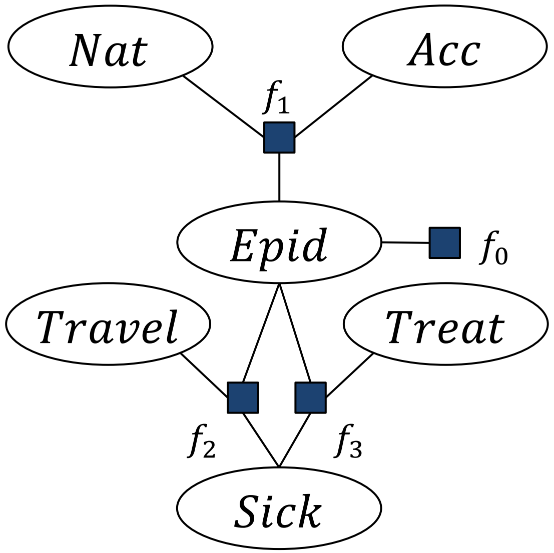
- Given model $F = \{f_i\}_{i=1}^{n}$ over random variables $\boldsymbol{R} = \{R_1, \dots, R_N\}$
  - $f_i = \phi_i(R_1, \dots, R_k)$

Semantics:

- Build full joint probability distribution P_F

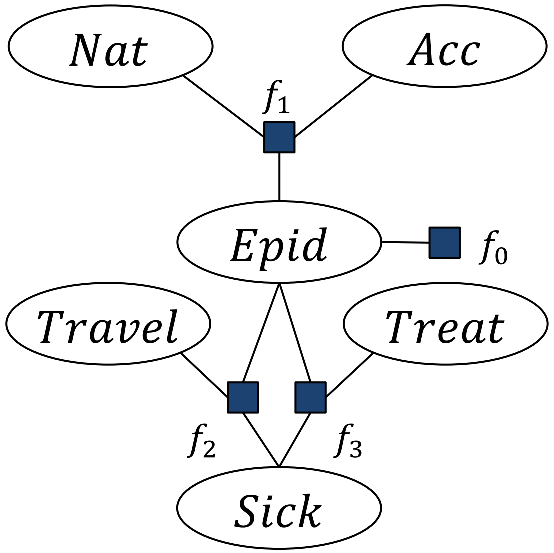
$$P_F = \frac{1}{Z} \prod_{i=1}^{n} \phi_i(R_1, \dots, R_k)$$

$$Z = \sum_{r_1 \in ran(R_1)} \sum_{r_n \in ran(R_n)} \prod_{i=1}^{n} \phi_i(r_1, \dots, r_k)$$



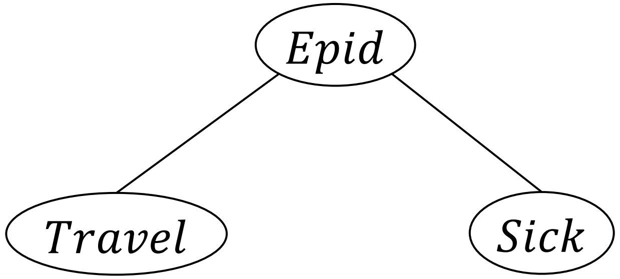
Model Representation: Factors

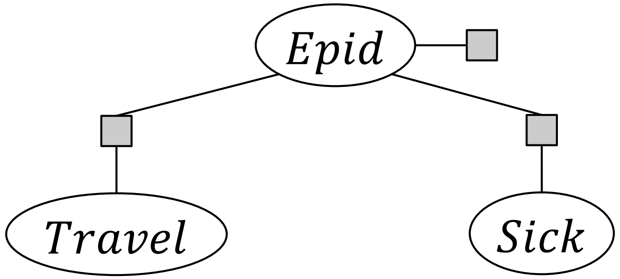
- Given model $F = \{f_i\}_{i=1}^{n}$ over random variables $\boldsymbol{R} = \{R_1, \dots, R_N\}$
  - $f_i = \phi_i(R_1, \dots, R_k)$
- Graphical representation: Factor graph (FG)
  - Each $R \in \boldsymbol{R}$: variable node in FG (ellipse)
  - Each $f \in F$: factor node in FG (box)
  - For each argument R in $f \in F$: edge between variable node for R and factor node for f



Other Model Representations

- Markov network (MN)
  - Alternative graphical representation of a factor-based model
  - Undirected graph
  - Factors: potential function for each clique in graph
    - If given a set of factors: add an edge between all random variables that occur together in a factor
    - With further information or the factors themselves, not clear what factors a model actually has just from an MN → disadvantage!
  - Semantics: Product of all factors, normalisation to get full joint
  - Neighbourhood directly defined between variables (not with factor nodes in between) → easier analysis (advantage)
- MNs and factor graphs have equivalent expressiveness

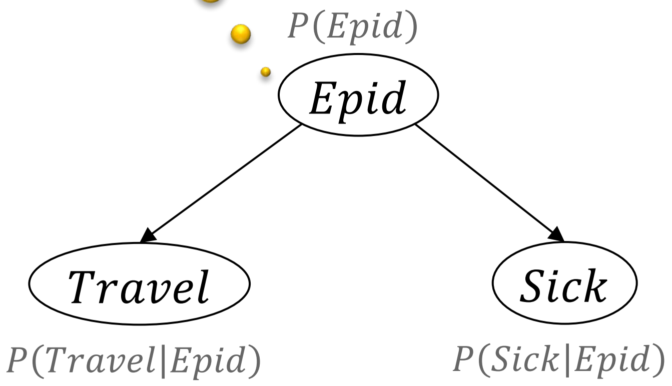


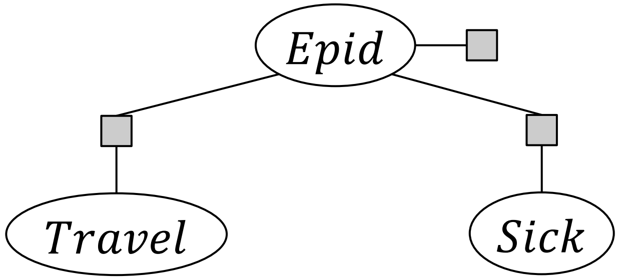


Other Model Representations

- Bayesian network (BN)
  - *Directed acyclic* graph
    - Explicit representation of (conditional) independences
    - Cannot model bidirectional influences → disadvantage!
  - Factors: set of probability distributions, one for each node
    - Prior probability tables for roots R: $P(R)$
    - Conditional probability tables (CPTs) for all other nodes N given its parents: $P(N|pa(N))$
  - Semantics
    - Product of tables, $Z = 1$ as tables are all probability distributions → advantage!
- Compared to undirected variants: independences readable in graph structure → advantage!

Which distributions does the BN have



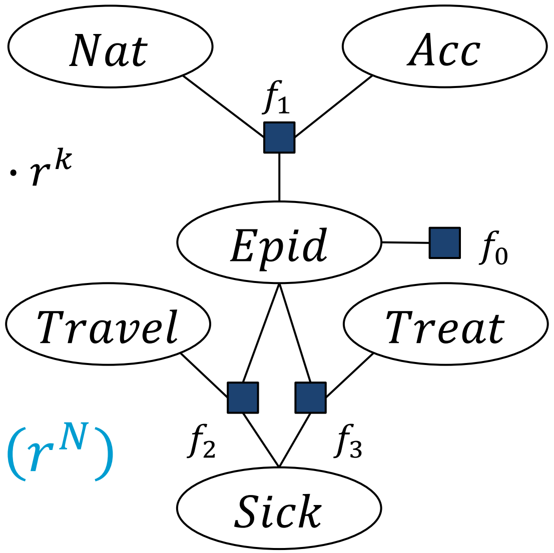


Space Complexity

- Given model $F = \{f_i\}_{i=1}^{n}$ over random variables $\boldsymbol{R}$
- Space complexity: $O(n \cdot r^k)$
  - $r = \max_{R \in \boldsymbol{R}} |ran(R)|$
  - $k = \max_{f \in F} |rv(f)|$
  - Derivation:

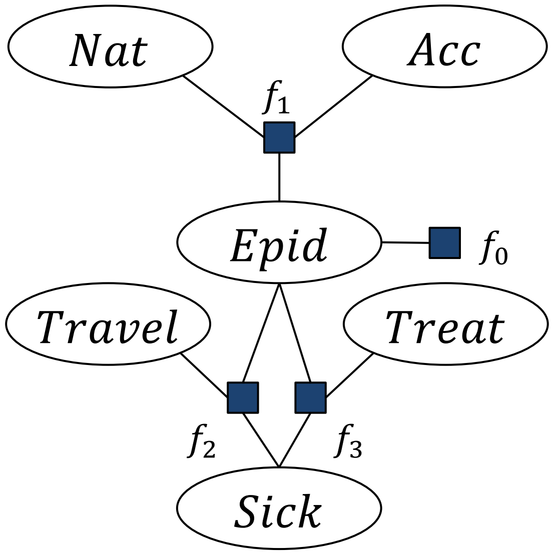
$$\sum_{f \in F} \prod_{R \in rv(f)} |ran(R)| \leq \sum_{f \in F} \prod_{R \in rv(f)} r = \sum_{f \in F} r^{|rv(f)|} \leq \sum_{f \in F} r^{\max_{f \in F} |rv(f)|} = \sum_{f \in F} r^k = n \cdot r^k$$

  - No longer exponential in $N = |\boldsymbol{R}|$, but in k
    - If $k \ll N$, n not depending exponentially on N: $O(n \cdot r^k) \ll O(r^N)$



Compact Encoding for Faster Inference

- Use factorisation for query answering
  - Sum out all non-query variables from a product
    - Distributive law holds
      → Move factors from the inner sums outwards if inputs not affected by sum
    - Sum out variable from smaller sub-products
    - Basic idea of variable elimination
- Focus for the remainder of the lecture: queries $P(\boldsymbol{S} \mid \boldsymbol{t})$
  - Evidence $\boldsymbol{t}$ = set of observations; may be empty:
    $\boldsymbol{t} = \emptyset \rightarrow P(\boldsymbol{S})$



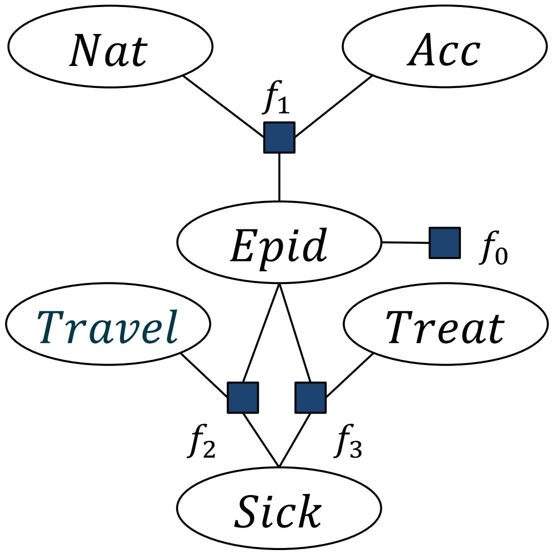
Variable Elimination (VE)

- Outline:
  1. Absorb evidence $\boldsymbol{t}$ in each factor covered by $\boldsymbol{t}$, i.e., $rv(f) \cap \boldsymbol{t} \neq \emptyset$,
  2. Sum out non-query variables $\boldsymbol{U} = \boldsymbol{R} \setminus rv(\boldsymbol{S}, \boldsymbol{t})$ using factorisation in model F

$$P(\boldsymbol{S} \mid \boldsymbol{t}) = \frac{1}{P(\boldsymbol{t})} \sum_{\boldsymbol{u} \in ran(\boldsymbol{U})} P_F(\boldsymbol{S}, \boldsymbol{t}, \boldsymbol{U} = \boldsymbol{u})$$

$$= \frac{1}{P(\boldsymbol{t})} \sum_{\boldsymbol{u} \in ran(\boldsymbol{U})} \prod_{f \in F} \phi_f \underbrace{(R_1, \ldots, R_k)}_{\pi_{rv(f)}(\boldsymbol{S}, \boldsymbol{t}, \boldsymbol{U} = \boldsymbol{u})}$$

  - Factor out factors from sums if arguments not covered by sum
  3. Divide by $P(\boldsymbol{t})$ = Normalise $P(\boldsymbol{S}, \boldsymbol{t})$
- Example: $P(Travel)$ in $F = \{f_i\}_{i=0}^{3}$



Variable Elimination (VE): Example

$P(Travel)$

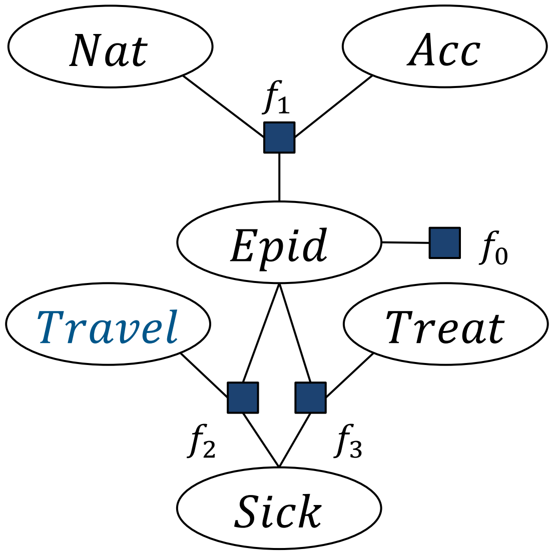
$$\propto \sum_{e \in \mathrm{Val}(E)} \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \sum_{s \in \mathrm{Val}(S)} \sum_{t \in \mathrm{Val}(T)} P_{\boldsymbol{R}}(E = e, N = n, A = a, S = s, Travel, T = t)$$

$$\propto \sum_{e \in \mathrm{Val}(E)} \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \sum_{s \in \mathrm{Val}(S)} \sum_{t \in \mathrm{Val}(T)} \prod_{i=0}^{3} \phi_i\,(\boldsymbol{R}_i = \boldsymbol{r}_i)$$

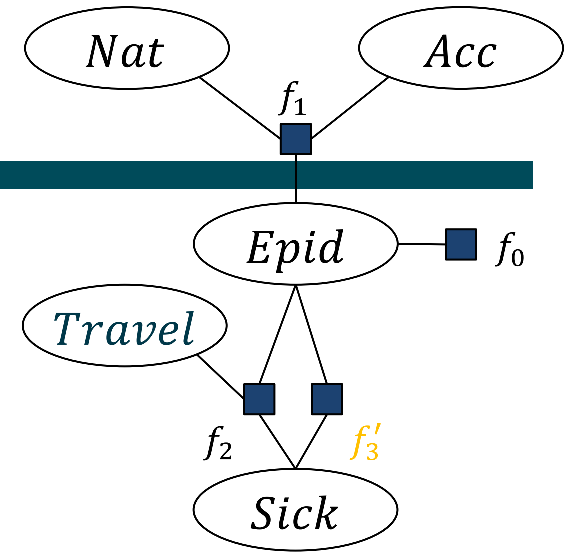
$$\propto \sum_{e \in \mathrm{Val}(E)} \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \sum_{s \in \mathrm{Val}(S)} \sum_{t \in \mathrm{Val}(T)} \phi_0(e)\phi_1(e,n,a)\phi_2(Travel,e,s)\phi_3(e,s,t)$$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a) \sum_{s \in \mathrm{Val}(S)} \phi_2(Travel,e,s) \sum_{t \in \mathrm{Val}(T)} \phi_3(e,s,t)$$

Sums can be computed independently → could be done in parallel



Variable Elimination (VE): Example

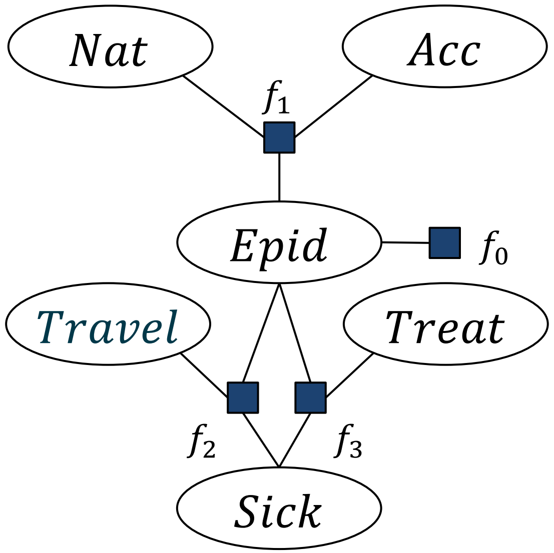


$$P(Travel) \propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a) \sum_{s \in \mathrm{Val}(S)} \phi_2(Travel,e,s) \sum_{t \in \mathrm{Val}(T)} \phi_3(e,s,t)$$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a) \sum_{s \in \mathrm{Val}(S)} \phi_2(Travel,e,s) \phi_3'(e,s)$$

$Epid$	$Sick$	$Treat$	ϕ_3
$false$	$false$	$false$	5
$false$	$false$	$true$	1
$false$	$true$	$false$	3
$false$	$true$	$true$	2
$true$	$false$	$false$	5
$true$	$false$	$true$	4
$true$	$true$	$false$	1
$true$	$true$	$true$	7

$Epid$	$Sick$	ϕ_3'
$false$	$false$	6
$false$	$true$	5
$true$	$false$	9
$true$	$true$	8



Variable Elimination (VE): Example

$$P(Travel) \propto \sum_{e\in Val(E)} \phi_0(e) \sum_{n\in Val(N)} \sum_{a\in Val(A)} \phi_1(e,n,a) \sum_{s\in Val(S)} \phi_2(Travel,e,s) \sum_{t\in Val(T)} \phi_3(e,s,t)$$

$$\propto \sum_{e\in Val(E)} \phi_0(e) \sum_{n\in Val(N)} \sum_{a\in Val(A)} \phi_1(e,n,a) \sum_{s\in Val(S)} \phi_2(Travel,e,s)\phi'_3(e,s)$$

$$\propto \sum_{e\in Val(E)} \phi_0(e) \sum_{n\in Val(N)} \sum_{a\in Val(A)} \phi_1(e,n,a) \sum_{s\in Val(S)} \phi_{23}(Travel,e,s)$$

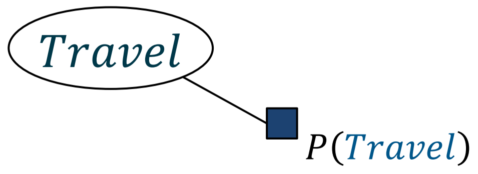
$$\propto \sum_{e\in Val(E)} \phi_0(e)\phi'_{23}(Travel,e) \sum_{n\in Val(N)} \sum_{a\in Val(A)} \phi_1(e,n,a)$$

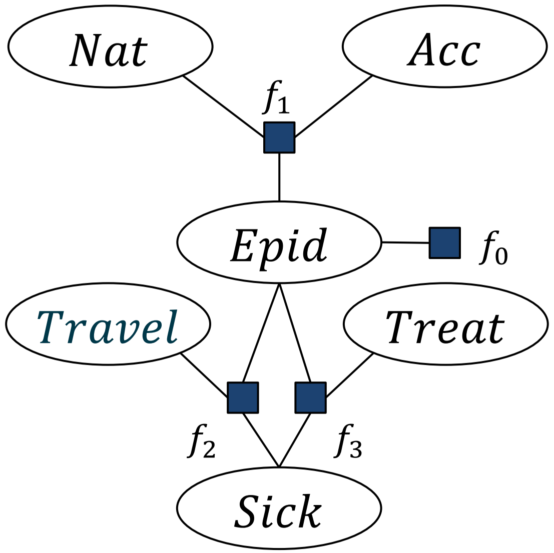
$$\propto \sum_{e\in Val(E)} \phi_0(e)\phi'_{23}(Travel,e)\phi''_1(e) = \sum_{e\in Val(E)} \phi(Travel,e)$$

$$= \phi'(Travel)$$

$$= \phi^n(Travel) = P(Travel)$$

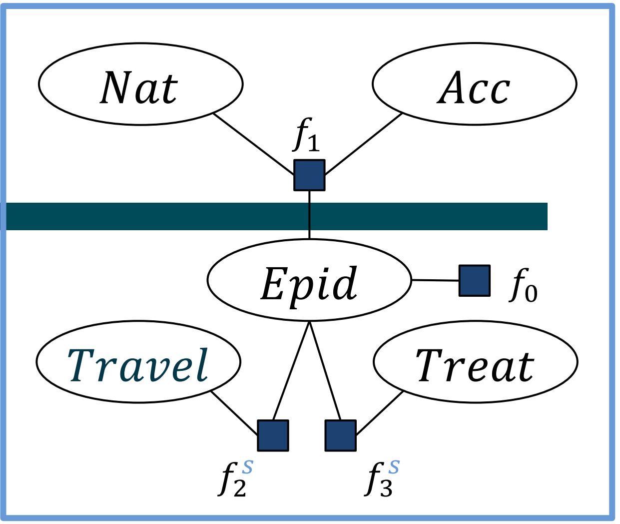
Intermediate results never larger than $2^3 < 2^6$





$E \triangleq Epid, N \triangleq Nat, A \triangleq Acc, S \triangleq Sick, T \triangleq Treat$

VE with Evidence



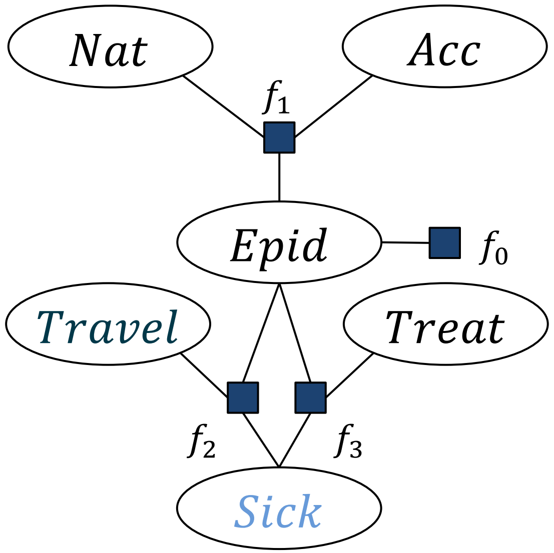
- Absorb each observation $t \in \boldsymbol{t}$ in each factor $f, t \cap rv(f) \neq \emptyset$
  1. Drop rows with $T \neq t$ → select all row with $T = t: f \leftarrow \sigma_{T=t}(f)$
  2. Drop T → project result onto $rv(f) \setminus \{T\}$: $f \leftarrow \pi_{rv(f)\setminus\{T\}}(f)$
  - Example: $P(Travel|sick)$ → f_2, f_3 have to absorb $sick$

$Epid$	$Sick$	$Treat$	ϕ_3
$false$	$false$	$false$	5
$false$	$false$	$true$	1
$false$	$true$	$false$	3
$false$	$true$	$true$	2
$true$	$false$	$false$	5
$true$	$false$	$true$	4
$true$	$true$	$false$	1
$true$	$true$	$true$	7

$Epid$	$Treat$	ϕ_3^s
$false$	$false$	3
$false$	$true$	2
$true$	$false$	1
$true$	$true$	7

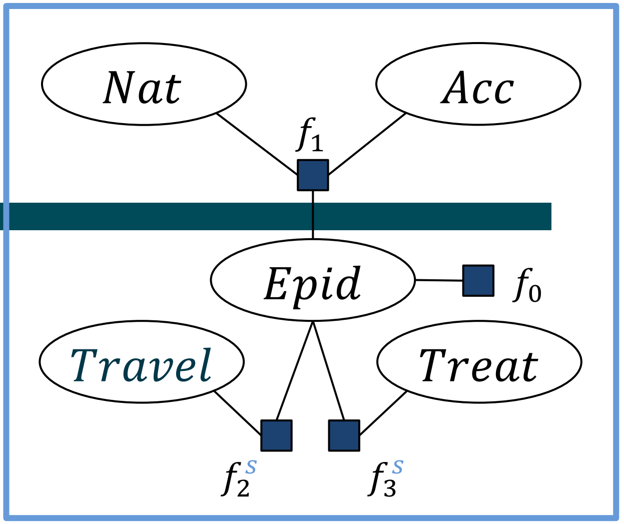
$Travel$	$Epid$	$Sick$	ϕ_2
$false$	$false$	$false$	20
$false$	$false$	$true$	24
$false$	$true$	$false$	5
$false$	$true$	$true$	6
$true$	$false$	$false$	28
$true$	$false$	$true$	8
$true$	$true$	$false$	7
$true$	$true$	$true$	2

$Travel$	$Epid$	ϕ_2^s
$false$	$false$	24
$false$	$true$	6
$true$	$false$	8
$true$	$true$	2



Model $F = \{f_0, f_1, f_2, f_3\}$ with query $P(Travel|sick)$ is turned into model $F' = \{f_0, f_1, f_2^s, f_3^s\}$ with query $P(Travel)$

VE with Evidence: Example



$P(Travel|sick)$

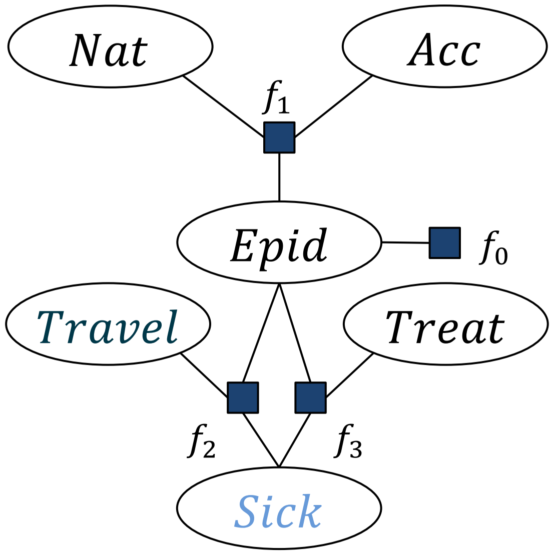
$$\propto \sum_{e \in \text{Val}(E)} \sum_{n \in \text{Val}(N)} \sum_{a \in \text{Val}(A)} \sum_{t \in \text{Val}(T)} P_{\boldsymbol{R}}(E = e, N = n, A = a, sick, Travel, T = t)$$

$$\propto \sum_{e \in \text{Val}(E)} \sum_{n \in \text{Val}(N)} \sum_{a \in \text{Val}(A)} \sum_{t \in \text{Val}(T)} \prod_{i=0}^{3} \phi_i\,(\boldsymbol{R}_i = \boldsymbol{r}_i)$$

$$\propto \sum_{e \in \text{Val}(E)} \sum_{n \in \text{Val}(N)} \sum_{a \in \text{Val}(A)} \sum_{t \in \text{Val}(T)} \phi_0(e)\phi_1(e,n,a)\phi_2(Travel, e, sick)\phi_3(e, sick, t)$$

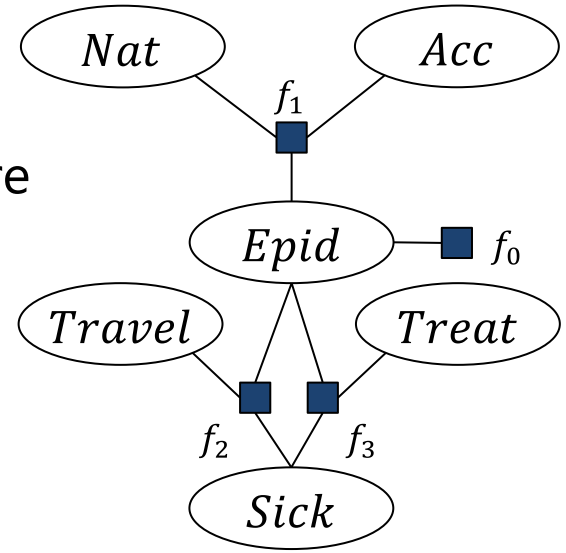
$$\propto \sum_{e \in \text{Val}(E)} \sum_{n \in \text{Val}(N)} \sum_{a \in \text{Val}(A)} \sum_{t \in \text{Val}(T)} \phi_0(e)\phi_1(e,n,a)\phi_2^s(Travel, e)\phi_3^s(e,t)$$

$$\propto \sum_{e \in \text{Val}(E)} \phi_0(e)\phi_2^s(Travel, e) \sum_{n \in \text{Val}(N)} \sum_{a \in \text{Val}(A)} \phi_1(e,n,a) \sum_{t \in \text{Val}(T)} \phi_3^s(e,t)$$



Elimination Order & Complexity

- Elimination order important
  - Wrong order → large intermediate result: consider eliminating $Epid$ first
  - Finding the best order not easier than inference using full joint
  - A lot of research has gone into finding a good order
- *Online greedy heuristic*: Variable R with smallest intermediate result
  - For each possible R to sum out
    - Collect all factors F_R containing R (would need to be multiplied before elimination)
    - Take number of arguments $|\mathrm{rv}(F_R)|$ (Intermediate result size before elimination)
  - Decision criterion: $\arg\min_R |\mathrm{rv}(F_R)|$ → one-step VE simulation
- Complexity depends exponentially on largest intermediate result:
  - $O(N \cdot r^w)$, w called *tree width*



Interim Summary

- Inference tasks: Answer query for (conditional) marginal probability (distribution)
  - Full joint: Exponential dependence on number of random variables (space, runtime) $\rightarrow O(r^N)$
- Factorised model
  - Use (conditional) independences for factorisation: $P(\boldsymbol{R}, \boldsymbol{S}|\boldsymbol{T}) = P(\boldsymbol{R}|\boldsymbol{T}) \cdot P(\boldsymbol{S}|\boldsymbol{T})$
    - Independence: $\boldsymbol{T} = \emptyset$
  - Model = set of factors
    - Factor graphs, Markov network, Bayesian network, Markov properties (briefly)
  - Reduces space complexity $\rightarrow O\left(n \cdot r^k\right)$
- Variable elimination (VE): inference algorithm to solve query answering problems
  - Absorb evidence, multiply factors, sum-out variables
  - Good elimination order required (heuristics), complexity possibly reduced to $O(N \cdot r^w)$

Overview: 2. Foundations

A. *Logic*
   - Propositional logic: alphabet, grammar, normal forms, rules
   - First-order logic: introducing quantifiers, domain constraints

B. *Probability theory*
   - Modelling: (conditional) probability distributions, random variables, marginal and joint distributions
   - Inference: axioms and basic rules, Bayes theorem, independence

C. *Probabilistic graphical models*
   - Syntax, semantics
   - Inference problems

→ Probabilistic Relational Models (PRMs)

Appendix

Formal Definitions of Absorption, Multiplication, Summing out, as well as VE

Full VE Example Calculations

Absorption: Formal Definition

- Operator: ABSORB
  - Inputs:
    - Factor $f = \phi(R_1, \dots, R_n) \in F$
    - Variable $R \in \{R_1, \dots, R_n\}$ at position i
    - Factor $f^r = \phi(R)$ with mappings $r \mapsto 1$ and $\forall r' \neq r \in ran(R) : r' \mapsto 0$ for observation $R = r$,
  - Precondition: *none*
  - Output: Factor $\phi'(R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n)$
    - For all possible valuations $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n$ of $R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n$
      - I.e., $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n \in ran(R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n)$, with

$$\phi'(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n) = \phi(r_1, \dots, r_{i-1}, r, r_{i+1}, \dots, r_n)$$

  - Postcondition: $F \cup \{f^r\} \sim F \setminus \{f\} \cup \{f^r, \text{ABSORB}(f, R, f^r)\}$

Factor Multiplication: Formal Definition

- Operator: MULTIPLY
  - Inputs:
    - Factor $f_1 = \phi_1(R_1, \dots, R_n) \in F$
    - Factor $f_2 = \phi_2(S_1, \dots, S_m) \in F$
  - Precondition: *none*
  - Output: Factor $\phi(T_1, \dots, T_k)$
    - $\{(T_1, \dots, T_k)\} = \{(R_1, \dots, R_n)\} \bowtie \{(S_1, \dots, S_m)\}$ (ordered union)
    - For all possible valuations $t_1, \dots, t_k$ of $T_1, \dots, T_k$, i.e., $t_1, \dots, t_k \in ran(T_1, \dots, T_k)$, with
      - $r_1, \dots, r_n = \pi_{R_1,\dots,R_n}(t_1, \dots, t_k)$ and $s_1, \dots, s_m = \pi_{S_1,\dots,S_m}(t_1, \dots, t_k)$ (select corresponding values from $t_1, \dots, t_k$)

$$\phi(t_1, \dots, t_k) = \phi_1(r_1, \dots, r_n) \cdot \phi_2(s_1, \dots, s_m)$$

  - Postcondition: $F \sim F \setminus \{f_1, f_2\} \cup \text{MULTIPLY}(f_1, f_2)$

Summing out Variables: Formal Definition

- Operator: SUM-OUT
  - Inputs:
    - Factor $f = \phi(R_1, \dots, R_n) \in F$
    - Variable $R \in \{R_1, \dots, R_n\}$ at position i to sum out
  - Precondition: $\forall f' \in F \setminus \{f\}: R \notin \mathrm{rv}(f')$
  - Output: Factor $\phi'(R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n)$
    - For each possible valuation $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n$ of $R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n$
      - I.e., $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n \in ran(R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n)$

$$\phi'(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n) = \sum_{r \in ran(R)} \phi(r_1, \dots, r_{i-1}, r, r_{i+1}, \dots, r_n)$$

  - Postcondition: $\sum_{r \in ran(R)} P_F \equiv P_{F \setminus \{f\} \cup \mathrm{SUM-OUT}(f,R)}$

VE Algorithm Using a Heuristics h

$\mathbf{VE}(F, \boldsymbol{S}, \{\phi_t(T_t)\}_{t=1}^{m}, h)$

Handle evidence

for $t = 1, \dots, m$ **do**

  while $\exists f \in F : T_t \in rv(f)$ **do** ▹ Absorb $\phi_t(T_t)$ in F

    $F \leftarrow F \setminus \{f\} \cup \{\text{ABSORB}(f, T, \phi_t(T))\}$

Eliminate non-query variables

while $\text{rv}(F) \setminus \boldsymbol{S} \neq \emptyset$ **do**

  $U \leftarrow \arg\min_R h(F)$ ▹ Choose next U to eliminate

  while $\exists f_1, f_2 \in F : U \in rv(f_1) \wedge rv(f_2)$ **do**

    $F \leftarrow F \setminus \{f_1, f_2\} \cup \{\text{MULTIPLY}(f_1, f_2)\}$ ▹ Multiply f_1, f_2 in F

  $F \leftarrow F \setminus \{f\} \cup \{\text{SUM-OUT}(f, U)\}$ ▹ Sum out U in F

while $\exists f_1, f_2 \in F$ **do** ▹ Multiply f_1, f_2 in F, until $|F| = 1$

  $F \leftarrow F \setminus \{f_1, f_2\} \cup \{\text{MULTIPLY}(f_1, f_2)\}$

Normalise the potentials in the one remaining $f \in F$ — Normalise

return f

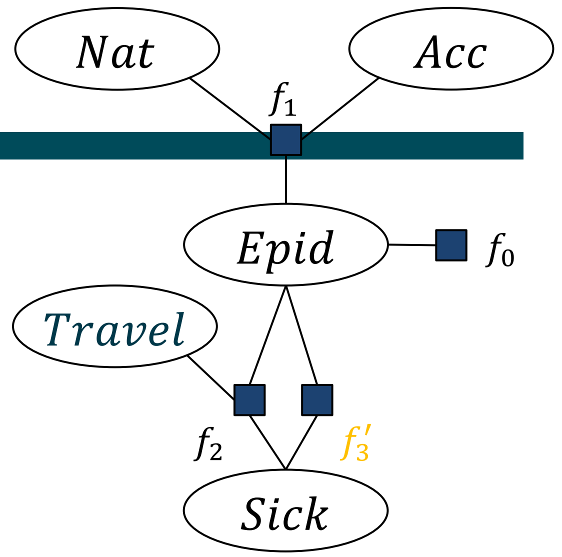
$P(Travel)$

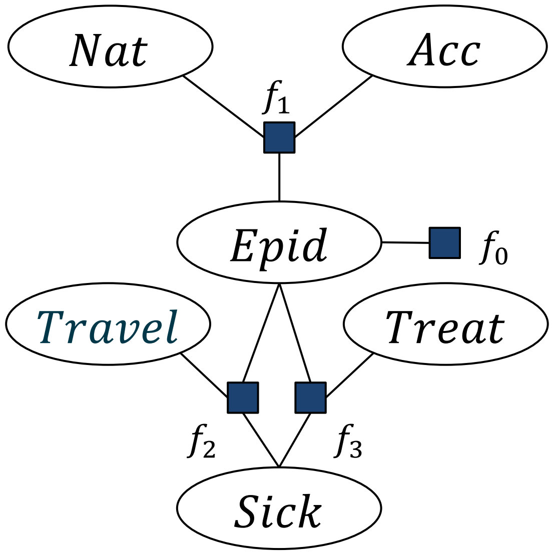
$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a) \sum_{s \in \mathrm{Val}(S)} \phi_2(Travel,e,s) \sum_{t \in \mathrm{Val}(T)} \phi_3(e,s,t)$$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a) \sum_{s \in \mathrm{Val}(S)} \phi_2(Travel,e,s) \phi_3'(e,s)$$

$Epid$	$Sick$	$Treat$	ϕ_3
$false$	$false$	$false$	5
$false$	$false$	$true$	1
$false$	$true$	$false$	3
$false$	$true$	$true$	2
$true$	$false$	$false$	5
$true$	$false$	$true$	4
$true$	$true$	$false$	1
$true$	$true$	$true$	7

$Epid$	$Sick$	ϕ_3'
$false$	$false$	6
$false$	$true$	5
$true$	$false$	9
$true$	$true$	8





$P(Travel)$

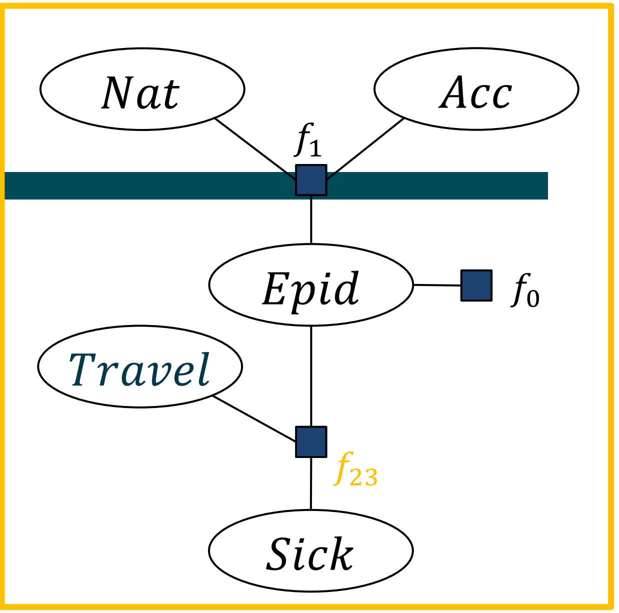
$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a) \sum_{s \in \mathrm{Val}(S)} \phi_2(Travel,e,s)\phi'_3(e,s)$$

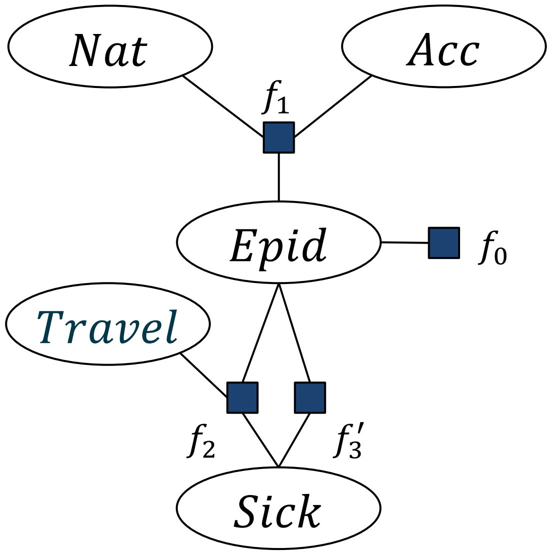
$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a) \sum_{s \in \mathrm{Val}(S)} \phi_{23}(Travel,e,s)$$

$Travel$	$Epid$	$Sick$	ϕ_2
$false$	$false$	$false$	20
$false$	$false$	$true$	24
$false$	$true$	$false$	5
$false$	$true$	$true$	6
$true$	$false$	$false$	28
$true$	$false$	$true$	8
$true$	$true$	$false$	7
$true$	$true$	$true$	2

$Epid$	$Sick$	ϕ'_3
$false$	$false$	6
$false$	$true$	5
$true$	$false$	9
$true$	$true$	8

$Travel$	$Epid$	$Sick$	ϕ_{23}
$false$	$false$	$false$	$20 \cdot 6 = 120$
$false$	$false$	$true$	$24 \cdot 5 = 120$
$false$	$true$	$false$	$5 \cdot 9 = 45$
$false$	$true$	$true$	$6 \cdot 8 = 48$
$true$	$false$	$false$	$28 \cdot 6 = 168$
$true$	$false$	$true$	$8 \cdot 5 = 40$
$true$	$true$	$false$	$7 \cdot 9 = 63$
$true$	$true$	$true$	$2 \cdot 8 = 16$





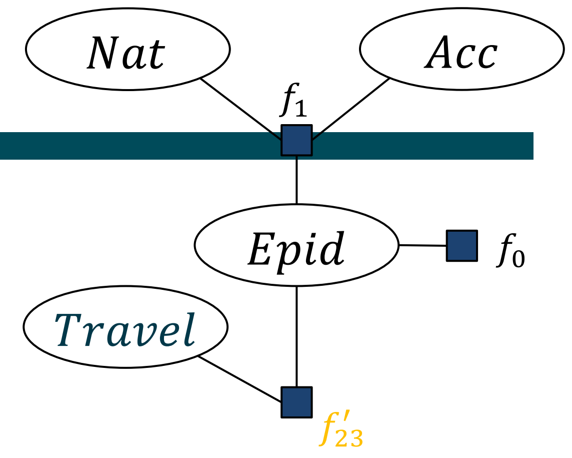
$P(Travel)$

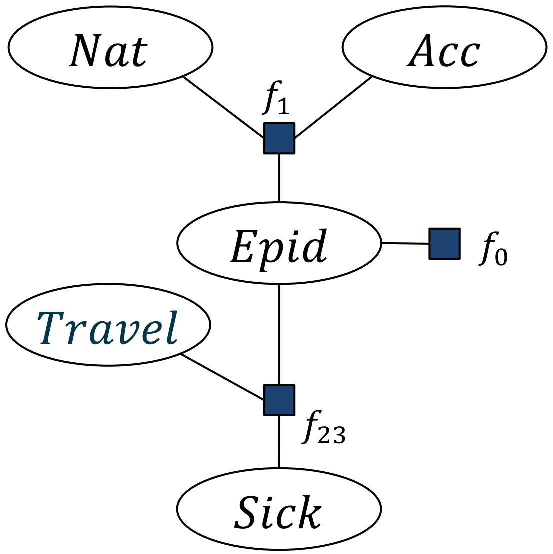
$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a) \sum_{s \in \mathrm{Val}(S)} \phi_{23}(Travel, e, s)$$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a)\, \phi'_{23}(Travel, e)$$

$Travel$	$Epid$	$Sick$	ϕ_{23}
$false$	$false$	$false$	120
$false$	$false$	$true$	120
$false$	$true$	$false$	45
$false$	$true$	$true$	48
$true$	$false$	$false$	168
$true$	$false$	$true$	40
$true$	$true$	$false$	63
$true$	$true$	$true$	16

$Travel$	$Epid$	ϕ'_{23}
$false$	$false$	240
$false$	$true$	93
$true$	$false$	208
$true$	$true$	79

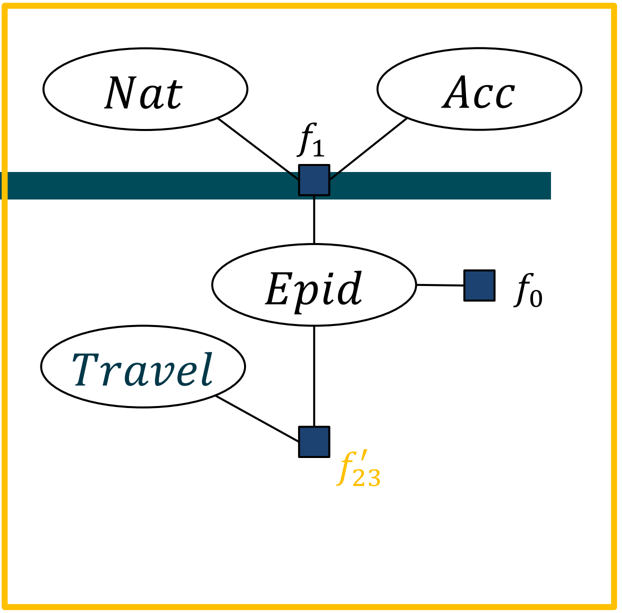


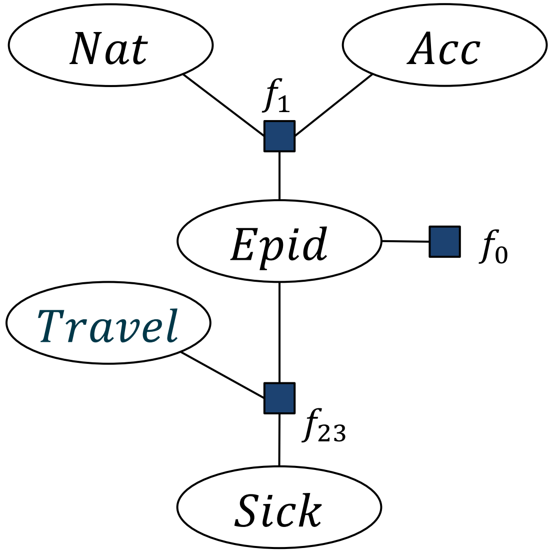


$P(Travel)$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a)\, \phi'_{23}(Travel, e)$$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e) \phi'_{23}(Travel, e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a)$$

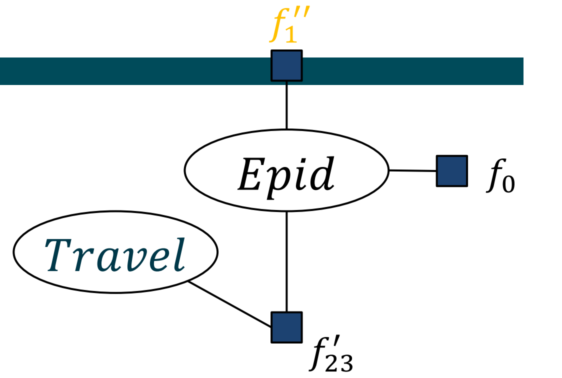




$P(Travel)$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e)\phi'_{23}(Travel,e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a)$$

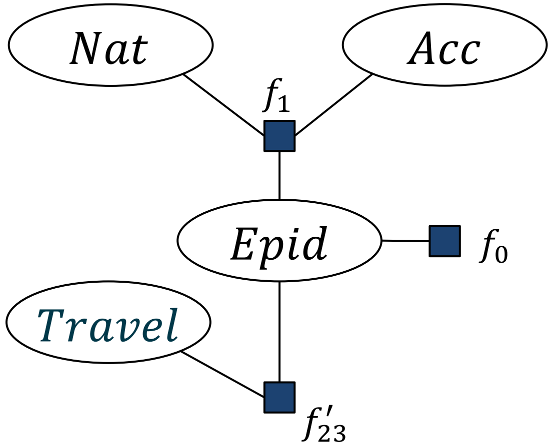
$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e)\phi'_{23}(Travel,e)\phi''_1(e)$$



Epid	*NatDis*	*Artif*	ϕ_1
false	*false*	*false*	12
false	*false*	*true*	2
false	*true*	*false*	3
false	*true*	*true*	1
true	*false*	*false*	7
true	*false*	*true*	4
true	*true*	*false*	5
true	*true*	*true*	1

Epid	*NatDis*	ϕ'_1
false	*false*	14
false	*true*	4
true	*false*	11
true	*true*	6

Epid	ϕ''_1
false	18
true	17



$P(Travel)$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e)\phi'_{23}(Travel, e)\phi''_1(e)$$

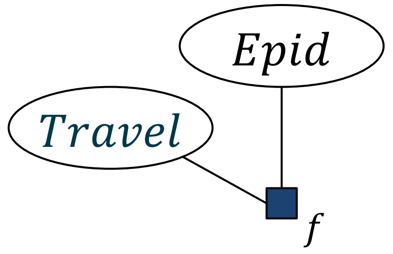
$$\propto \sum_{e \in \mathrm{Val}(E)} \phi(Travel, e)$$

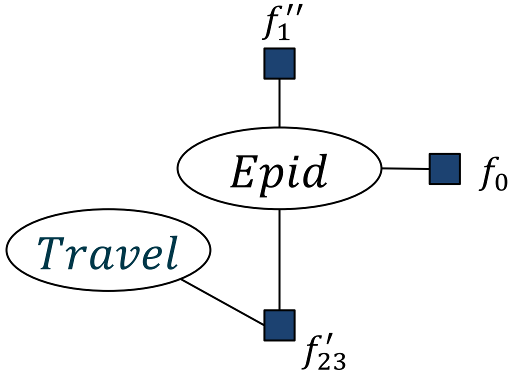
$Travel$	$Epid$	ϕ'_{23}
$false$	$false$	240
$false$	$true$	93
$true$	$false$	208
$true$	$true$	79

$Epid$	ϕ''_1
$false$	18
$true$	17

$Epid$	ϕ_0
$false$	50
$true$	1

$Travel$	$Epid$	ϕ
$false$	$false$	$240 \cdot 18 \cdot 50 = 216{,}000$
$false$	$true$	$93 \cdot 17 \cdot 1 = 1{,}581$
$true$	$false$	$208 \cdot 18 \cdot 50 = 187{,}200$
$true$	$true$	$79 \cdot 17 \cdot 1 = 1{,}343$





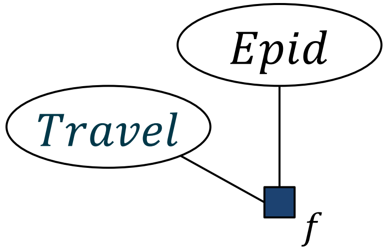
$P(Travel)$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi(Travel, e)$$

$$\propto \phi'(Travel)$$

$Travel$	$Epid$	ϕ
$false$	$false$	216,000
$false$	$true$	1,581
$true$	$false$	187,200
$true$	$true$	1,343

$Travel$	ϕ'
$false$	217,581
$true$	188,543



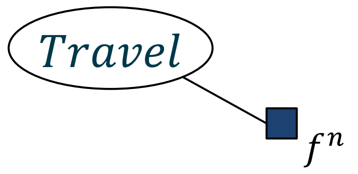
$P(Travel)$

$\propto \phi'(Travel)$

$= \phi^n(Travel)$
$= P(Travel)$

$Travel$	ϕ
$false$	217,581
$true$	188,543

$Travel$	ϕ^n
$false$	$\frac{217{,}581}{217{,}581 + 188{,}543} = \frac{217{,}581}{406{,}124} = 0.54$
$true$	$\frac{188{,}543}{217{,}581 + 188{,}543} = \frac{188{,}543}{406{,}124} = 0.46$



VE with Evidence: Example

$P(Travel \mid sick)$

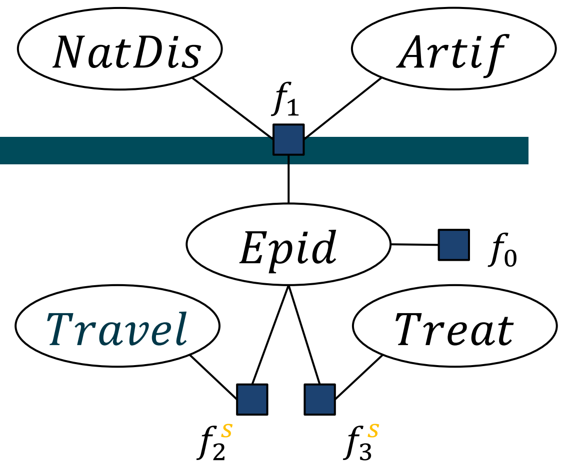
$$\propto \sum_{e \in \mathrm{Val}(E)} \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \sum_{t \in \mathrm{Val}(T)} P_{\boldsymbol{R}}(E = e, N = n, A = a, sick, Travel, T = t)$$

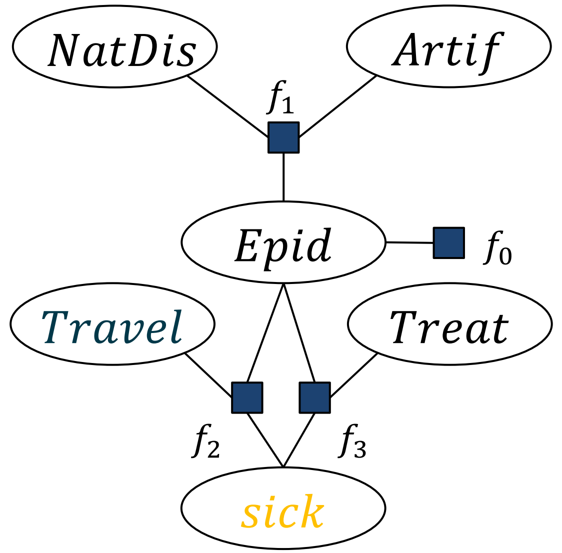
$$\propto \sum_{e \in \mathrm{Val}(E)} \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \sum_{t \in \mathrm{Val}(T)} \prod_{i=0}^{3} \phi_i(\boldsymbol{R}_i = \boldsymbol{r}_i)$$

$$\propto \sum_{e \in \mathrm{Val}(E)} \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \sum_{t \in \mathrm{Val}(T)} \phi_0(e)\phi_1(e,n,a)\phi_2(Travel, e, sick)\phi_3(e, sick, t)$$

$$\propto \sum_{e \in \mathrm{Val}(E)} \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \sum_{t \in \mathrm{Val}(T)} \phi_0(e)\phi_1(e,n,a)\phi_2^s(Travel, e)\phi_3^s(e, t)$$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e)\phi_2^s(Travel, e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a) \sum_{t \in \mathrm{Val}(T)} \phi_3^s(e, t)$$





VE with Evidence: Example

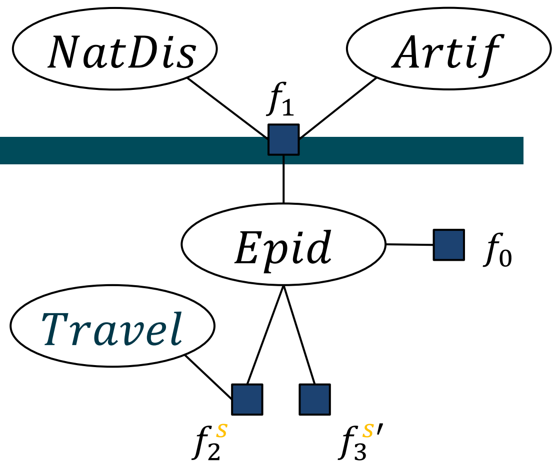
$P(Travel \mid sick)$

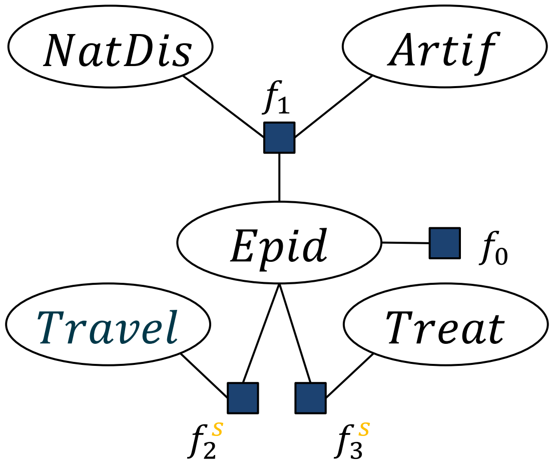
$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e)\phi_2^s(Travel, e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a) \sum_{t \in \mathrm{Val}(T)} \phi_3^s(e,t)$$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e)\phi_2^s(Travel, e)\phi_3^{s\prime}(e) \sum_{n \in \mathrm{Val}(N)} \sum_{a \in \mathrm{Val}(A)} \phi_1(e,n,a)$$

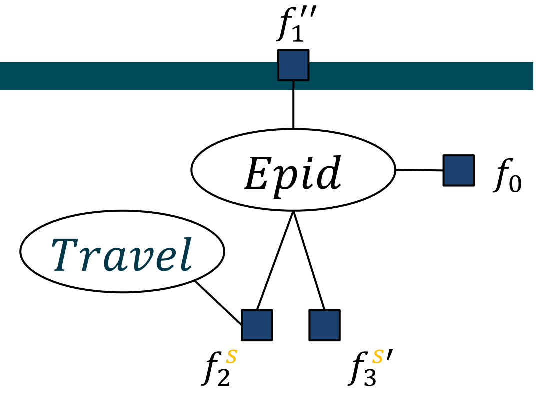
$Epid$	$Treat$	ϕ_3^s
$false$	$false$	3
$false$	$true$	2
$true$	$false$	1
$true$	$true$	7

$Epid$	$\phi_3^{s\prime}$
$false$	5
$true$	8





VE with Evidence: Example



$P(Travel \mid sick)$

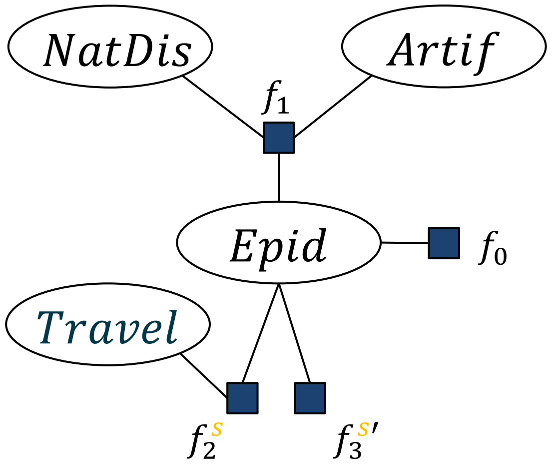
Wie vorher

$$\propto \sum_{e \in Val(E)} \phi_0(e)\phi_2^s(Travel, e)\phi_3^{s\prime}(e) \sum_{n \in Val(N)} \sum_{a \in Val(A)} \phi_1(e, n, a)$$

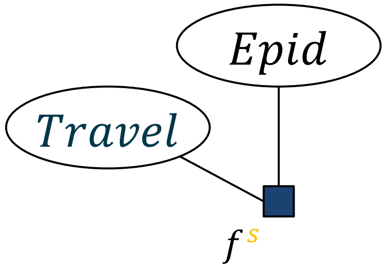
$$\propto \sum_{e \in Val(E)} \phi_0(e)\phi_2^s(Travel, e)\phi_3^{s\prime}(e)\phi_1(e'')$$

$Epid$	$NatDis$	$Artif$	ϕ_1
$false$	$false$	$false$	12
$false$	$false$	$true$	2
$false$	$true$	$false$	3
$false$	$true$	$true$	1
$true$	$false$	$false$	7
$true$	$false$	$true$	4
$true$	$true$	$false$	5
$true$	$true$	$true$	1

$Epid$	ϕ_1''
$false$	18
$true$	17



VE with Evidence: Example



$P(Travel \mid sick)$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi_0(e)\phi_2^s(Travel, e)\phi_3^{s\prime}(e)\phi_1^{\prime\prime}(e)$$

$$\propto \sum_{e \in \mathrm{Val}(E)} \phi^s(Travel, e)$$

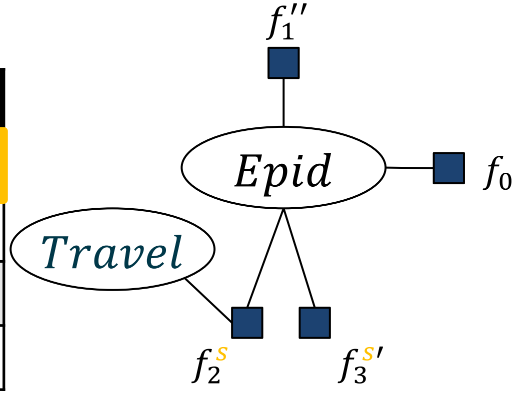
$Travel$	$Epid$	ϕ_2^s
$false$	$false$	24
$false$	$true$	6
$true$	$false$	8
$true$	$true$	2

$Epid$	ϕ_0
$false$	50
$true$	1

$Epid$	$\phi_3^{s\prime}$
$false$	5
$true$	8

$Epid$	$\phi_1^{\prime\prime}$
$false$	18
$true$	17

$Travel$	$Epid$	ϕ
$false$	$false$	$24 \cdot 50 \cdot 5 \cdot 18 = 108.000$
$false$	$true$	$6 \cdot 1 \cdot 8 \cdot 17 = 816$
$true$	$false$	$8 \cdot 50 \cdot 5 \cdot 18 = 36.000$
$true$	$true$	$2 \cdot 1 \cdot 8 \cdot 17 = 272$



VE with Evidence: Example

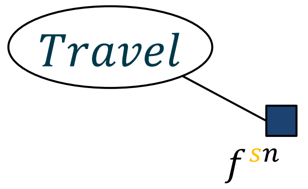
$P(Travel \mid sick)$

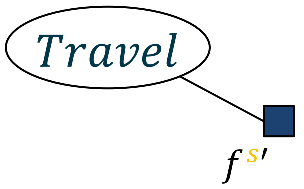
$\propto \sum_{e \in \mathrm{Val}(E)} \phi^{s}(Travel, e)$

$\propto \phi^{s'}(Travel)$

$= \phi^{sn}(Travel)$

$= P(Travel \mid sick)$





$Travel$	$Epid$	ϕ^{s}
$false$	$false$	108.000
$false$	$true$	816
$true$	$false$	36.000
$true$	$true$	272

$Travel$	$\phi^{s'}$
$false$	108.816
$true$	36.272

$Travel$	ϕ^{sn}
$false$	$\frac{108.816}{108.816 + 36.272} = \frac{108.816}{145.088} = 0.75$
$true$	$\frac{36.272}{108.816 + 36.272} = \frac{36.272}{145.088} = 0.25$