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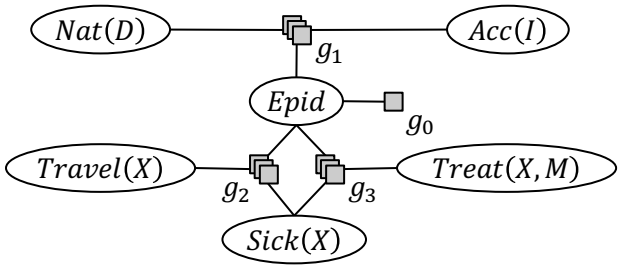
INSTITUT FÜR INFORMATIONSSYSTEME

Dynamic Probabilistic Relational Models

Probabilistic Relational Models (PRMs)



$10\ Presents(X, P, C) \Rightarrow Attends(X, C)$

$3.75\ Publishes(X, C) \wedge FarAway(C) \Rightarrow Attends(X, C)$

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Contents

Outline: 2. Probabilistic Relational Models (PRMs)

A. Parfactor models (PMs)

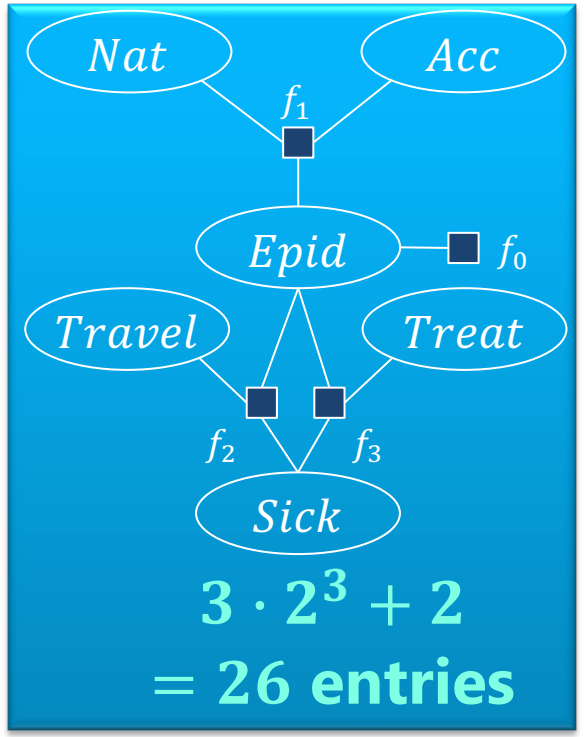
- Motivation: Symmetries and relations
- Syntax, semantics
- Graphical representation
- Inference tasks

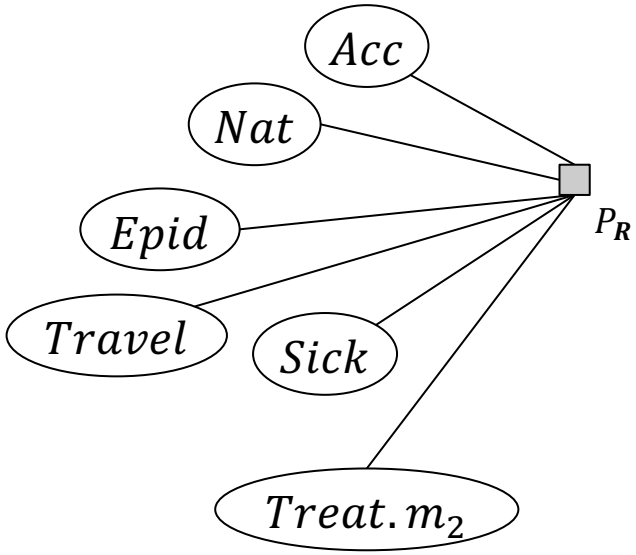
B. Markov logic networks (MLNs)

- Syntax, semantics of MLNs
- Graphical representation
- Turning MLNs into PMs and vice versa

Exponential Blowup! → Sparse Encoding

Acc	*Nat*	*Treat*	*Epid*	*Travel*	*Sick*	*P*
false	*false*	*false*	*false*	*false*	*false*	0.025
false	*false*	*false*	*false*	*false*	*true*	0.009
false	*false*	*false*	*false*	*true*	*false*	
false	*false*	*false*	*false*	*true*		
false	*false*	*false*	*true*			
false	*false*	*false*				
false	*false*					
false						

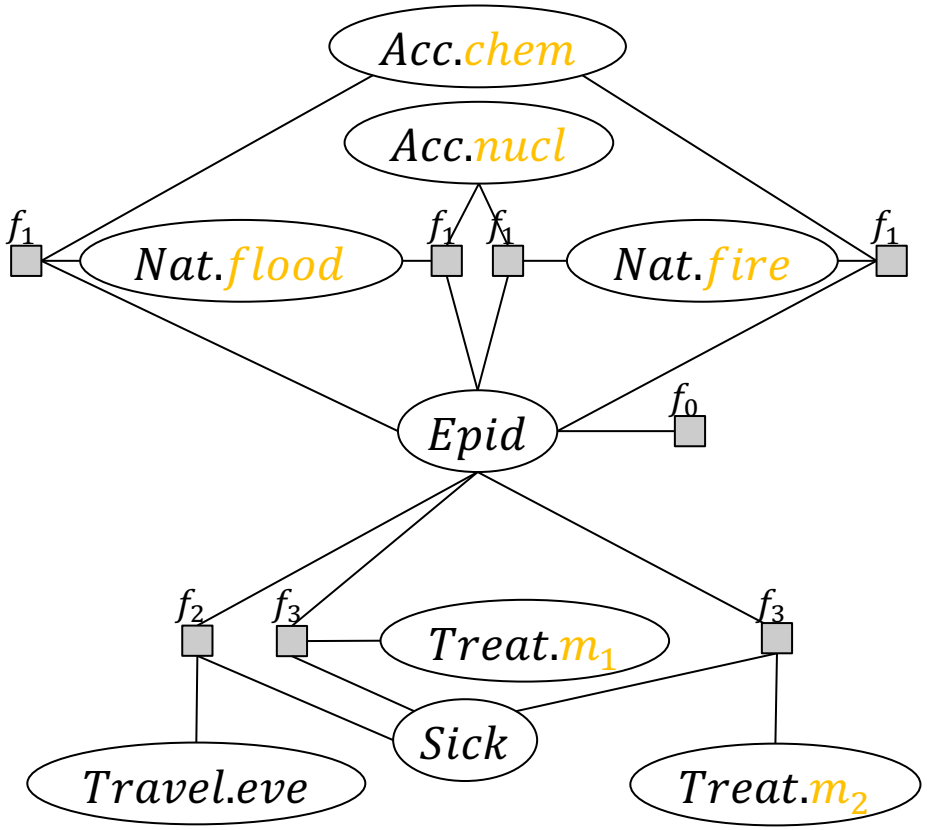




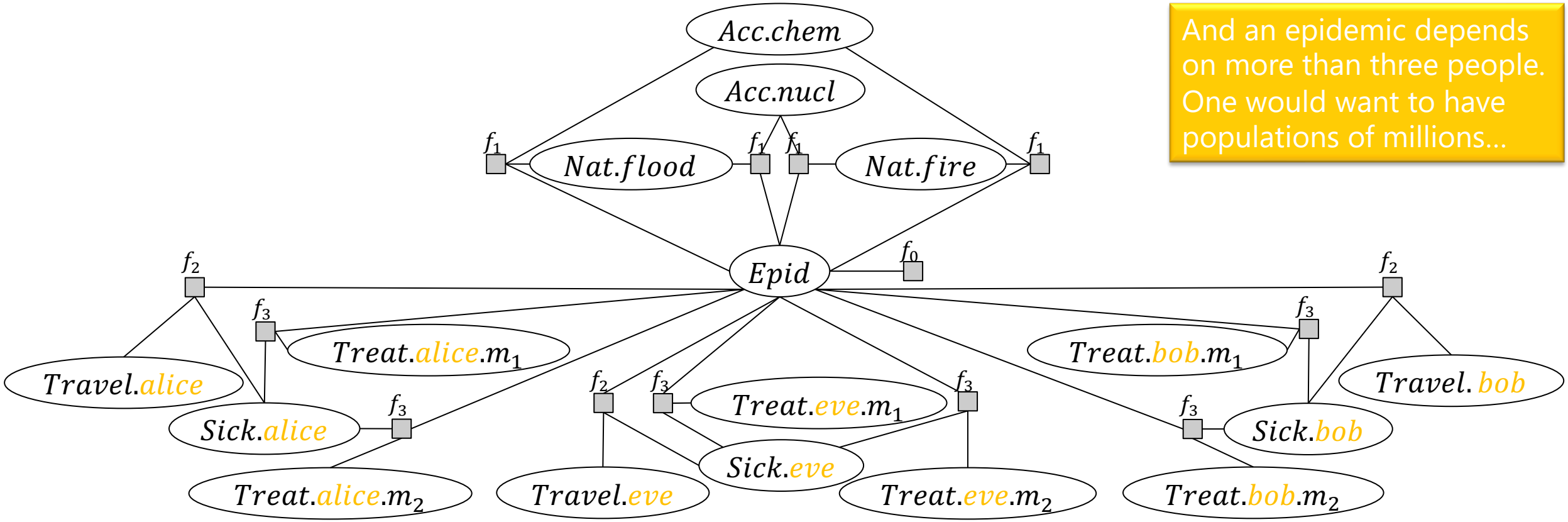
$2^6 = 64$ possible worlds

Adding relations means adding *Sick*, *Treat*, *Travel* variables for each person, blowing up the model further

Problem: Adding Objects and Relations in Propositional Formalisms Clunky…



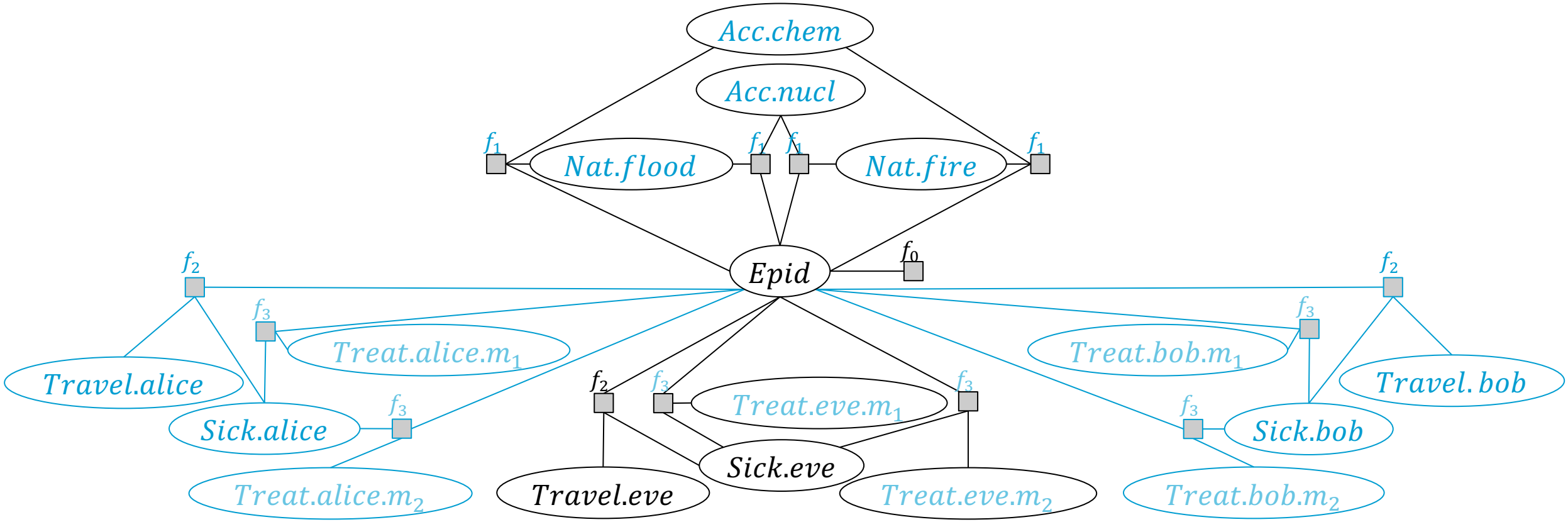
Problem: … and Models Explode with Them



And an epidemic depends on more than three people. One would want to have populations of millions…

$13 \cdot 2^3 + 2 = 104$ entries in 14 factors, 17 variables

Propositional → First-order View



Symmetries in graph / relations in scenario

Propositional → First-order View

- From probabilistic perspective
- From logic perspective

- Instead of *propositional* constraints, we have *first-order* constraints:

$$\frac{epid \Rightarrow sick}{epid \Rightarrow sick.eve}$$

$$met.eve.alice \wedge sick.eve \Rightarrow sick.alice$$
$$met.alice.eve \wedge sick.alice \Rightarrow sick.eve$$
$$\vdots$$
$$\forall x : epid \Rightarrow sick(x)$$
$$\forall x\big(\exists y : met(x,y) \wedge sick(x) \Rightarrow sick(y)\big)$$

- But probabilistic perspective allows soft constraints with a probability assigned

- Add *logical variables* to random variables

Epid	*Sick*	*P*
false	*false*	0.9
false	*true*	0.1
true	*false*	0.6
true	*true*	0.4

Epid	*Sick.eve*	ϕ
false	*false*	
⋮	⋮	

Epid	*Sick*(X)	ϕ
false	*false*	9
false	*true*	1
true	*false*	6
true	*true*	4

Encoding Relations and Symmetries

- Logical variables to encode
  - Recurring structures in graphs
  - Relational structures in data
- By parameterising random variables with logical variables
- And using those parameterised random variables (PRVs) as inputs to factors
  - Forming parametric factors (parfactors)
- Allowing to encode
  - Duplicate factors only once
  - That factors apply to multiple individuals (constants)

David Poole: First-order Probabilistic Inference. In *IJCAI-03 Proceedings of the 18th International Joint Conference on Artificial Intelligence*, 2003.
Brian Milch, Luke S. Zettelmoyer, Kristian Kersting, Michael Haimes, and Leslie Pack Kaelbling. Lifted Probabilistic Inference with Counting Formulas AAAI-08 Proceedings of the 23rd Conference on Artificial Intelligence, 2008.
Nima Taghipour: Lifted Probabilistic Inference by Variable Elimination. KU Leuven, 2013.
Tanya B: Rescued from a Sea of Queries: Exact Inference in Probabilistic Relational Models. UzL, 2020.

Notations

- Variable: capitalised character/string
  - Propositional random variable: speaking names or if abstract: R, R_1, R_i (default)
  - PRV: speaking names or if abstract: A, A_1, A_i (default)
  - Logical variable: X (default)
  - Value: lowercase character/string of corresponding variable
- Set: **boldface**
  - $\boldsymbol{V} = \{V_1, \dots, V_n\}, \boldsymbol{v} = \{v_1, \dots, v_n\}$
  - Exception to the rule: model (set of factors) F, relational model (set of parfactors) G
- Sequence: *calligraphy-face*
  - $\mathcal{V} = (V_1, \dots, V_n), \mathcal{v} = (v_1, \dots, v_n)$
  - In abuse of notation, we use set relations ($\in, \subseteq, \supseteq$) and operations ($\cap, \cup, \setminus$) on sequences, assuming a matching order or if not given, a reasonable reordering

Logical Variables, Domains, Universe

- Logical variable L
  - to encode patterns or relations (first-order constructs)
- Possible values (constants) of logical variables = domain $dom(L) = \{c_1, \dots, c_k\}$
- E.g.,
  - Logical variables D, M, I, X with domains
    - $dom(D) = \{fire, flood\}$
    - $dom(M) = \{m_1, m_2\}$
    - $dom(I) = \{chem, nucl\}$
    - $dom(X) = \{alice, eve, bob\}$
- Universe $\boldsymbol{D} = \{c_1, \dots, c_K\}$
  - All constants occurring in a scenario, i.e., $dom(L) \subseteq \boldsymbol{D}$

Constraints

- To restrict applicability of domain constants: Constraint $C = (\mathcal{X}, C_{\mathcal{X}})$
  - $\mathcal{X} = (X_1, \ldots, X_n)$ a sequence of logical variables
  - $C_{\mathcal{X}} \subseteq \times_{i=1}^{n} dom(X_i)$ a subset of domain combinations
    - Set of sequences of constants in the order given by $\mathcal{X}$
  - If no restrictions apply, i.e., $C_{\mathcal{X}} = \times_{i=1}^{n} dom(X_i)$: $C = \top$
    - Can be omitted
- E.g.,
  - $(X, \{(alice), (eve)\})$
  - $((X, M),$

$$\{\ (alice, m_1), (alice, m_2),\ (eve, m_1), (eve, m_2),\ (bob, m_1), (bob, m_2)\ \}) = \top$$

Constraints = Abstraction of a database

The constraint apparatus appears overwhelming but it just tells you for which constants something applies

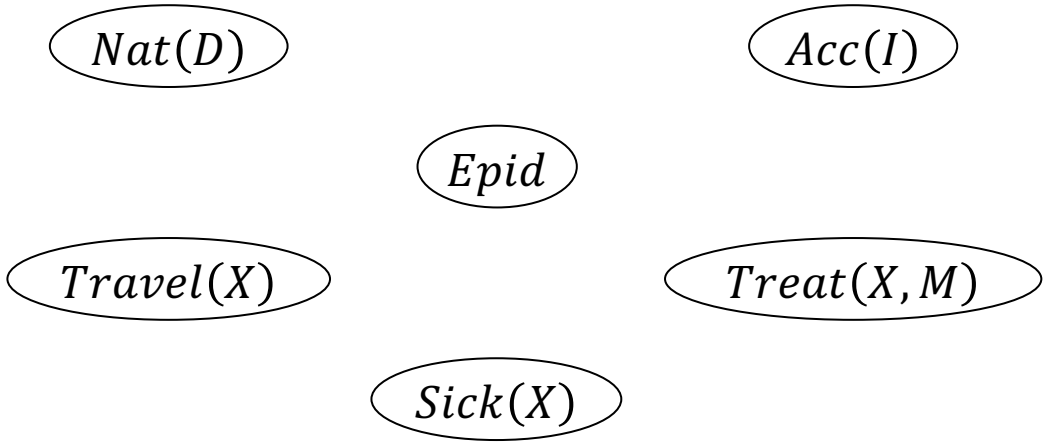
- Necessary for proofs but not for general understanding

PRVs

- Random variable R parameterised with logical variables $X_1, \ldots, X_n, n \geq 0$ = PRV

$$A = R(X_1, \ldots, X_n)$$

  - if $n = 0$, $A = R$ a propositional random variable
- Range as before: $ran(A) = \{v_1, \ldots, v_m\}$
- E.g.,
  - $Treat(X, M)$ with $n = 2$
  - $Epid$ with $n = 0$
  - $Treat(X, M), Travel(X)$, $Sick(X)$ sharing the same logical variable



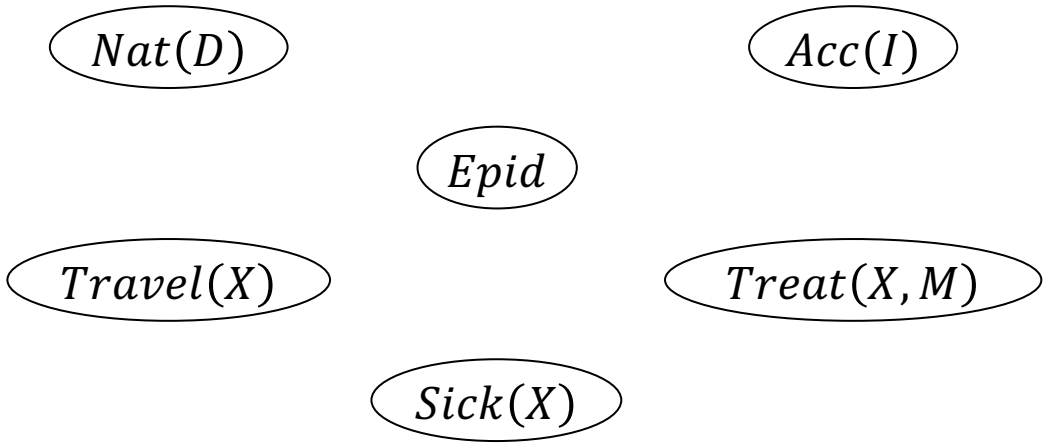
Restricting PRVs

- PRV A under constraint C written as $A_{|C}$
  - $C = (\mathcal{X}, C_{\mathcal{X}})$, $\mathcal{X} \supseteq lv(A)$
    - $lv(.)$ referring to the logical variables of the given input
  - Represents a set of propositional random variables
- E.g.,
  - $Sick(X)_{|C}$ with $C = (X, \{(alice), (eve)\})$
    - Represents $Sick(alice), Sick(eve)$
  - $Treat(X, M)_{|\mathsf{T}}$ with $\mathsf{T} = \big((X, M), dom(X) \times dom(M)\big)$
    - Represents $Treat(alice, m_1), Treat(alice, m_2), Treat(eve, m_1), Treat(eve, m_2), Treat(bob, m_1), Treat(bob, m_2)$

It may happen that a $C = (\mathcal{X}, C_{\mathcal{X}})$ talks about more logical variables $\mathcal{X}$ than the thing it applies to with logical variables $\mathcal{Y} \subset \mathcal{X}$

- To restrict C to $\mathcal{Y}$, we use the projection operation π from relational algebra:

if $\mathcal{Y} \subseteq \mathcal{X}, \pi_{\mathcal{Y}}(C) := \big(\mathcal{Y}, \pi_{\mathcal{Y}}(C_{\mathcal{X}})\big)$



Parfactors

- Factor with PRVs as arguments = parfactor

  $g = \forall \boldsymbol{x} \in C_{\mathcal{X}} : \phi(\mathcal{A}\theta_{\boldsymbol{x}})_{|(\mathcal{X},C_{\mathcal{X}})}$

  - $\mathcal{A} = (A_1, \ldots, A_k)$ sequence of PRVs containing logical variables $\mathcal{X}$
  - ϕ as before:

    $\phi: \times_{i=1}^{k} ran(A_i) \longrightarrow \mathbb{R}^{0,+}$

    - At least one potential > 0
  - Shorthand: $\phi(\mathcal{A})_{|C}$
    - If $C = \top$: $\phi(\mathcal{A})$
  - E.g., $g_2 =$

    $\forall \boldsymbol{x} \in C_{\mathcal{X}}: \phi\big(Travel(x), Epid, Sick(x)\big)_{|((X),\{(alice),(eve),(bob)\})}$

    $= \phi\big(Travel(X), Epid, Sick(X)\big)_{|\top}$

    $= \phi\big(Travel(X), Epid, Sick(X)\big)$

$Travel(X)$	$Epid$	$Sick(X)$	ϕ_2
$false$	$false$	$false$	5
$false$	$false$	$true$	0
$false$	$true$	$false$	4
$false$	$true$	$true$	6
$true$	$false$	$false$	4
$true$	$false$	$true$	6
$true$	$true$	$false$	2
$true$	$true$	$true$	9

Grounding of Parfactors

- Grounding of parfactor $gr(g)$
  - Obtain set of factors $\{f_i\}_{i=1}^{m}$ that g represents
  - Replace logical variables in g with constants in constraint of g:

$$gr\left(\phi(\mathcal{A})_{|(X,C_X)}\right) = \bigcup_{\theta_x \in \boldsymbol{\theta}} \phi(\mathcal{A}\theta_x)$$

$$\boldsymbol{\theta} = \bigcup_{\boldsymbol{x} \in C_X} \left\{ \bigcup_{x \in \boldsymbol{x}} \{X \to x\} \right\}$$

    - $\boldsymbol{\theta}$ set of all possible substitutions $\theta_{\boldsymbol{x}}$ to apply to $\mathcal{X}$ in $\mathcal{A}$
  - E.g., $gr(g_2) = \{f_2^1, f_2^2, f_2^3\}$
    - $g_2 = \phi\left(Travel(X), Epid, Sick(X)\right)_{|\mathrm{T}}$

$Travel(X)$	$Epid$	$Sick(X)$	ϕ_2
$false$	$false$	$false$	5
$false$	$false$	$true$	0
$false$	$true$	$false$	4
$false$	$true$	$true$	6
$true$	$false$	$false$	4
$true$	$false$	$true$	6
$true$	$true$	$false$	2
$true$	$true$	$true$	9

Grounding: Example

- E.g., $gr(g_2) = \{f_2^1, f_2^2, f_2^3\}$

$Travel(alice)$	$Travel(eve)$	$Epid$	$Sick(eve)$	ϕ_2^1
$false$	$false$	$false$	$false$	5
$false$	$false$	$false$	$true$	0
$false$	$false$	$true$	$false$	4
$false$	$false$	$true$	$true$	6
$true$	$true$	$false$	$false$	4
$true$	$true$	$false$	$true$	6
$true$	$true$	$true$	$false$	2
$true$	$true$	$true$	$true$	9

$Epid$	$Sick(bob)$	ϕ_2^3
$false$	$false$	5
$false$	$true$	0
$true$	$false$	4
$true$	$true$	6
$false$	$false$	4
$false$	$true$	6
$true$	$false$	2
$true$	$true$	9

=

$Travel(X)$	$Epid$	$Sick(X)$	ϕ_2
$false$	$false$	$false$	5
$false$	$false$	$true$	0
$false$	$true$	$false$	4
$false$	$true$	$true$	6
$true$	$false$	$false$	4
$true$	$false$	$true$	6
$true$	$true$	$false$	2
$true$	$true$	$true$	9

Symmetries Within

- Assume four epidemics with identical characteristics
  - $Epid_1, Epid_2, Epid_3, Epid_4$
  - Reasonable to model the epidemics such that it does not matter which $Epid$ variables specifically are $true$ or $false$, i.e., they are interchangeable

- All $false$ maps to 8
- 1 $true$, 3 $false$ maps to 6
- 2 $true$, 2 $false$ maps to 4
- 3 $true$, 1 $false$ maps to 2
- All $true$ maps to 0

→ Five lines enough to describe

$\#_{true}$	$\#_{false}$	ϕ
[0,4]		8
[1,3]		6
[2,2]		4
[3,1]		2
[4,0]		0

$Epid_1$	$Epid_2$	$Epid_3$	$Epid_4$	ϕ
$false$	$false$	$false$	$false$	8
$false$	$false$	$false$	$true$	6
$false$	$false$	$true$	$false$	6
$false$	$false$	$true$	$true$	4
$false$	$true$	$false$	$false$	6
$false$	$true$	$false$	$true$	4
$false$	$true$	$true$	$false$	4
$false$	$true$	$true$	$true$	2
$true$	$false$	$false$	$false$	6
$true$	$false$	$false$	$true$	4
$true$	$false$	$true$	$false$	4
$true$	$false$	$true$	$true$	2
$true$	$true$	$false$	$false$	4
$true$	$true$	$false$	$true$	2
$true$	$true$	$true$	$false$	2
$true$	$true$	$true$	$true$	0

Counting Random Variable

- New PRV type: (Parameterised) counting random variable ((P)CRV) $\#_X[A_{|C}]$
  - $A_{|C}$ a PRV under constraint C
  - $X \in lv(A)$
  - Range values: Histogram $h = \{(v_i, n_i)\}_{i=1}^{m}$
    - $m = |ran(A)|$ (number of buckets)
    - $n = \sum_{i=1}^{m} n_i = \left|gr\left(A_{|\pi_X(C)}\right)\right|$ (number of instances to distribute into buckets)
    - $v_i \in ran(A)$ (buckets)
    - $n_i \in \mathbb{N}$ (number of instances in bucket v_i)
    - Shorthand: $[n_1, \ldots, n_m]$
- Range of a (P)CRV = space of histograms fulfilling the conditions on the histograms
  - (All possible ways of distributing n interchangeable instances into m buckets)
- Single histogram encodes several interchangeable assignments at once
  - Given by multinomial coefficient $Mul(h)$

$$Mul(h) = \frac{n!}{\prod_{i=1}^{m} n_i!}$$

  - If $m = 2$, binomial coefficient:

$$\binom{n}{n_1} = \frac{n!}{(n-n_1)!\,n_1!} = \frac{n!}{n_2!\,n_1!}$$

CRV: Example

- (P)CRV $\#_X[A_{|C}]$
  - Range values: Histogram $h = \{(v_i, n_i)\}_{i=1}^{m}$
    - $m = |ran(A)|$ (number of buckets)
    - $n = \sum_{i=1}^{m} n_i = |gr(A_{|C})|$ (number of instances to distribute into buckets)
    - $v_i \in ran(A)$ (buckets)
    - $n_i \in \mathbb{N}$ (number of instances in bucket v_i)
    - Shorthand: $[n_1, \dots, n_m]$
  - Single histogram encodes several interchangeable assignments at once:

$$Mul(h) = \frac{n!}{\prod_{i=1}^{m} n_i!}$$

- E.g., CRV: $\#_E[Epid(E)]$
  - $ran(Epid(E)) = \{true, false\} \rightarrow m = 2$
  - $dom(E) = \{e_1, e_2, e_3, e_4\} \rightarrow n = 4$
  - Range values and multiplicities

$$\{(true, 0), (false, 4)\} = [0,4] \quad Mul([0,4]) = \frac{4!}{0! \cdot 4!} = 1$$
$$\{(true, 1), (false, 3)\} = [1,3] \quad Mul([1,3]) = \frac{4!}{1! \cdot 3!} = 4$$
$$\{(true, 2), (false, 2)\} = [2,2] \quad Mul([2,2]) = \frac{4!}{2! \cdot 2!} = 6$$
$$\{(true, 3), (false, 1)\} = [3,1] \quad Mul([3,1]) = \frac{4!}{3! \cdot 1!} = 4$$
$$\{(true, 4), (false, 0)\} = [4,0] \quad Mul([4,0]) = \frac{4!}{4! \cdot 0!} = 1$$

CRV: Example

- E.g., (continued)
  - CRV: $\#_E[Epid(E)]$
    - Range values
      $[0,4], [1,3], [2,2], [3,1], [4,0]$
      $1 \quad 4 \quad 6 \quad 4 \quad 1$
      how many assignments encoded
  - $g' = \phi(\#_E[Epid(E)])$

$\#_E[Epid(E)]$	ϕ'
[0,4]	8
[1,3]	6
[2,2]	4
[3,1]	2
[4,0]	0

$Epid_1$	$Epid_2$	$Epid_3$	$Epid_4$	ϕ
false	false	false	false	8
false	false	false	true	6
false	false	true	false	6
false	false	true	true	4
false	true	false	false	6
false	true	false	true	4
false	true	true	false	4
false	true	true	true	2
true	false	false	false	6
true	false	false	true	4
true	false	true	false	4
true	false	true	true	2
true	true	false	false	4
true	true	false	true	2
true	true	true	false	2
true	true	true	true	0

CRVs Continued

- (P)CRV $\#_X\left[A_{|C}\right]$ with
  - $m = |ran(A)|$ (number of buckets)
  - $n = \sum_{i=1}^{m} n_i = \left|gr\left(A_{|\pi_X(C)}\right)\right|$ (number of instances to distribute into buckets)
- Instead of m^n mappings in the ground factor, the counted factor has

$$\binom{n+m-1}{n-1}$$

mappings

  - Upper bound of range size of a CRV:

$$\binom{n+m-1}{n-1} \leq n^m$$

Exponential in number of random variables n

Polynomial in number of random variables n

CRVs Continued

- Counting binds a logical variable, i.e.,

$$lv\left(\#_X\left[A_{|C}\right]\right) = lv(A) \setminus \{X\}$$

  - Constants that X represents made explicit through counts in histograms
  - Compare the examples on the right
    - $\phi\left(Epid, Sick(X)\right)$ is a representative for the factors behind all groundings of X
    - $\phi'(\#_E[Epid(E)])$ already references all groundings of E in histograms

$Epid$	$Sick(X)$	ϕ
$false$	$false$	9
$false$	$true$	1
$true$	$false$	6
$true$	$true$	4

$\#_E[Epid(E)]$	ϕ'
[0,4]	8
[1,3]	6
[2,2]	4
[3,1]	2
[4,0]	0

- Restriction:
  Only one logical variable can be counted at a time in a PRV!
  - In the current formalisation
    (it is possible to define more complex counting operations)

Parfactors Revisited

- Factor with PRVs as arguments = parfactors

  $g = \forall \boldsymbol{x} \in C_X : \phi(\mathcal{A}\theta_{\boldsymbol{x}})_{|(X,C_X)}$

  - $\mathcal{A} = (A_1, \ldots, A_k)$ sequence of P(C)RVs containing logical variable
  - ϕ as before:

    $\phi: \times_{i=1}^{k} ran(A_i) \longrightarrow \mathbb{R}^{0,+}$

    - At least one potential > 0
  - Shorthand $\phi(\mathcal{A})_{|C}$
    - If $C = \top$: $\phi(\mathcal{A})$
  - E.g.,

$\mathcal{A}$ may contain ground random variables, PRVs, CRVs, PCRVs

$$g_2 = \phi\big(Travel(X), Epid, Sick(X)\big)$$
$$g' = \phi(\#_E[Epid(E)])$$
$$g'' = \phi(Sick(X), \#_E[Epid(E)])$$

$Sick(X)$	$\#_{\boldsymbol{E}}[Epid(E)]$	ϕ''
$false$	[0,4]	9
$false$	[1,3]	7
$false$	[2,2]	5
$false$	[3,1]	3
$false$	[4,0]	4
$true$	[0,4]	5
$true$	[1,3]	4
$true$	[2,2]	3
$true$	[3,1]	2
$true$	[4,0]	6

Parfactor Model

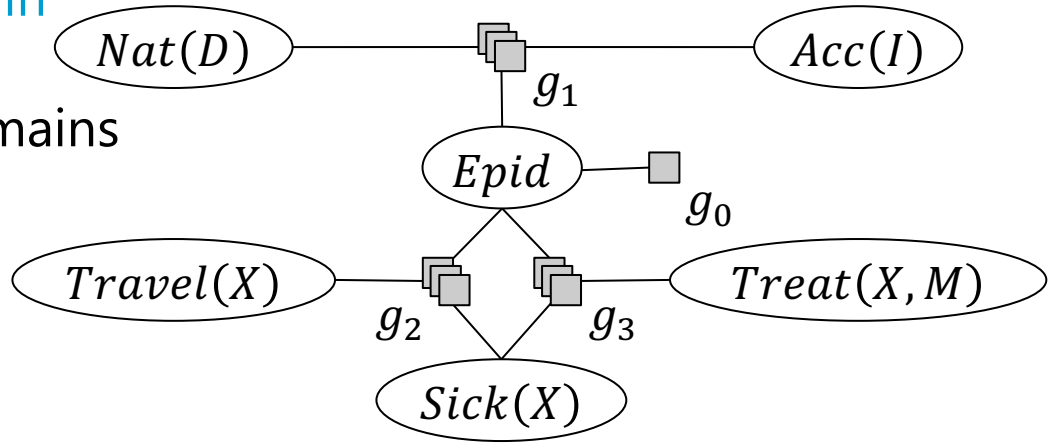
- Set of parfactors = model

$$G = \{g_i\}_{i=1}^{n}$$

  - Given the definitions of PRVs, it is possible to form a parfactor model that is completely propositional
    - If no logical variables used or domains of size 1
  - E.g., $G = \{g_i\}_{i=0}^{3}$
    - Plus full specifications of potential functions
    - $\mathbf{3 \cdot 2^3 + 2 = 26}$ **entries** independent of domain sizes!
    - Instead of $\mathbf{13 \cdot 2^3 + 2 = 106}$ **entries** with domains

$$dom(D) = \{fire, flood\}, |dom(D)| = 2$$
$$dom(M) = \{m_1, m_2\}, |dom(M)| = 2$$
$$dom(I) = \{chem, nucl\}, |dom(I)| = 2$$
$$dom(X) = \{alice, eve, bob\}, |dom(D)| = 3$$

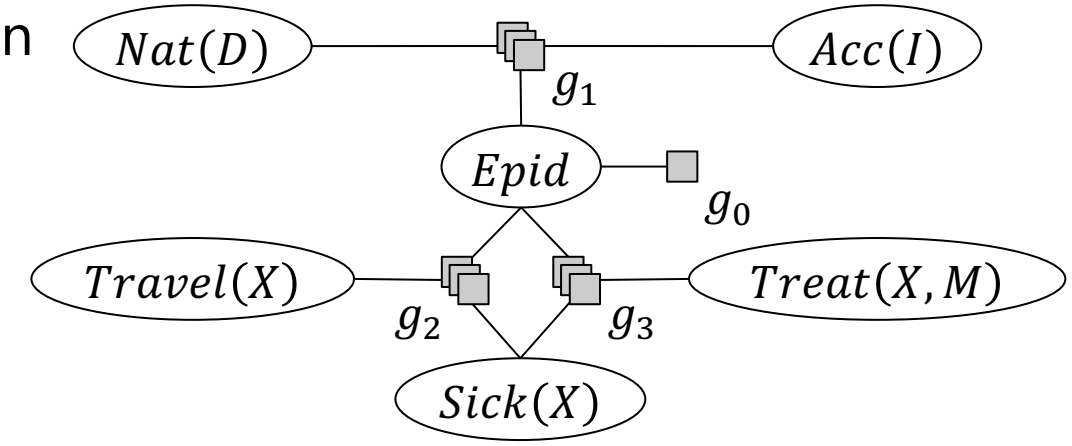
$$g_0 = \phi(Epid)$$
$$g_1 = \phi(Epid, Nat(D), Acc(I))$$
$$g_2 = \phi(Travel(X), Epid, Sick(X))$$
$$g_3 = \phi(Epid, Sick(X), Treat(X, M))$$



Parfactor Graphs

- Graphical representation of $G = \{g_i\}_{i=1}^{n}$: Parfactor graph
  - Analogous to factor graph
  - Each $A \in rv(G)$: variable node in graph (ellipse)
  - Each $g \in G$: factor node in graph (box)
    - Layered if $lv(g) \neq \emptyset$
  - For each argument A in $g \in G$: edge between variable node for A and factor node for g
  - Constraints are not depicted
  - Example
    - Six variable nodes
    - Four factor nodes, three of them layered

$$g_0 = \phi(Epid)$$
$$g_1 = \phi\big(Epid, Nat(D), Acc(I)\big)$$
$$g_2 = \phi\big(Travel(X), Epid, Sick(X)\big)$$
$$g_3 = \phi\big(Epid, Sick(X), Treat(X, M)\big)$$

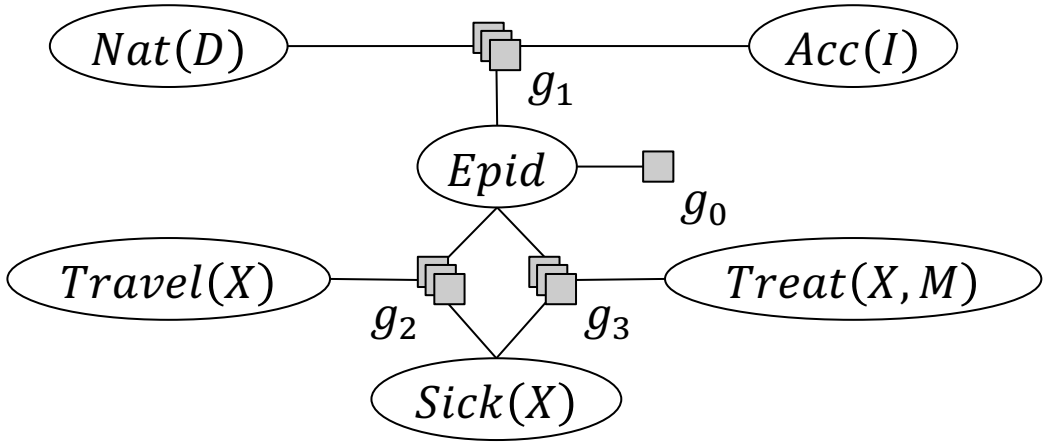


Grounding of Models

- Grounding of model G

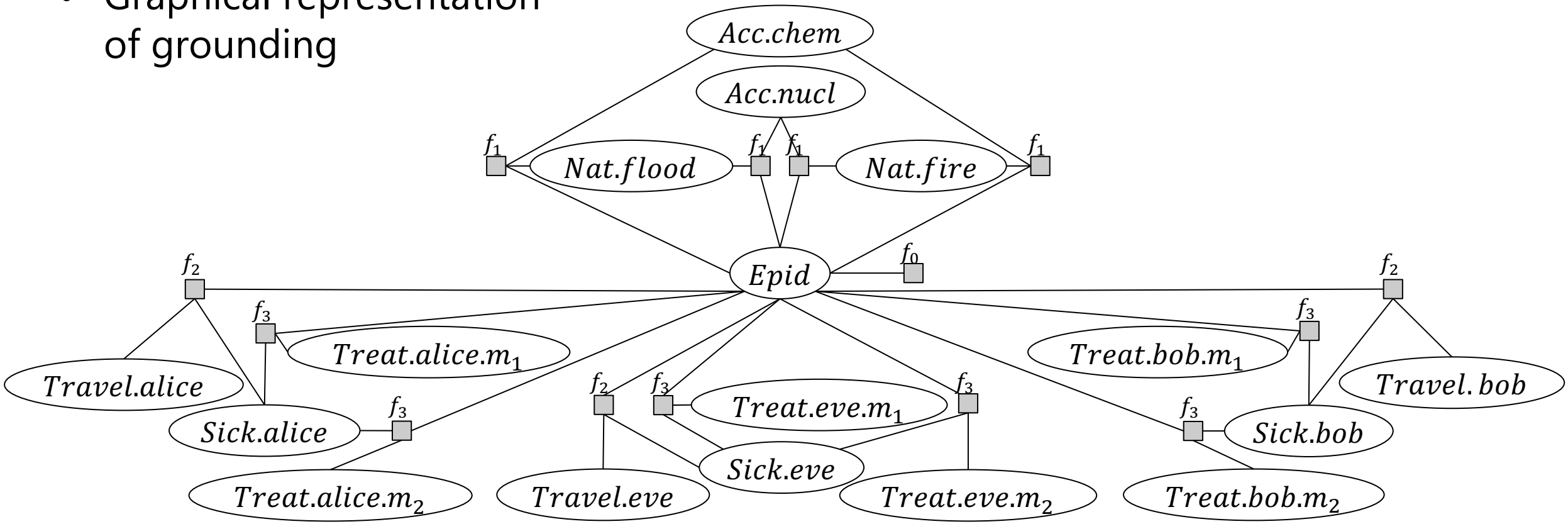
$$gr(G) = \bigcup_{g \in G} gr(g)$$

  - Depends on constraints and thus, domains
  - E.g., example model with T constraints and domains
    - $dom(D) = \{fire, flood\}$
    - $dom(I) = \{chem, nucl\}$
    - $dom(M) = \{m_1, m_2\}$
    - $dom(X) = \{alice, eve, bob\}$



Grounding of Models

- Graphical representation of grounding



Semantics

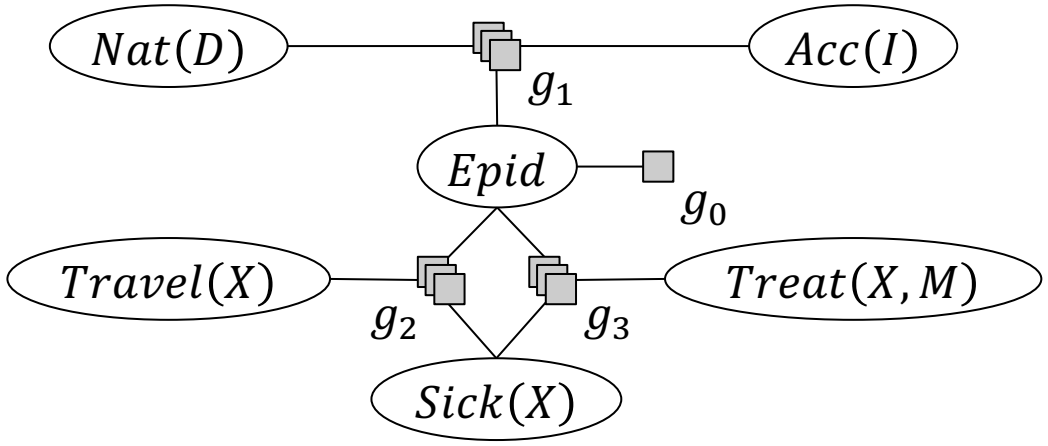
- Given model $G = \{g_i\}_{i=1}^{n}$
  - Over PRVs $rv(G)$
  - Equivalent propositional model $F = gr(G)$ over random variables $gr\big(rv(G)\big) = \{R_1, \ldots, R_N\}$ with semantics $P_{\boldsymbol{R}}$
- Semantics: Full joint probability distribution P_G

$$P_G = \frac{1}{Z} \prod_{f \in gr(G)} f$$

$$Z = \sum_{r_1 \in ran(R_1)} \ldots \sum_{r_N \in ran(R_N)} \prod_{f \in gr(G)} f$$

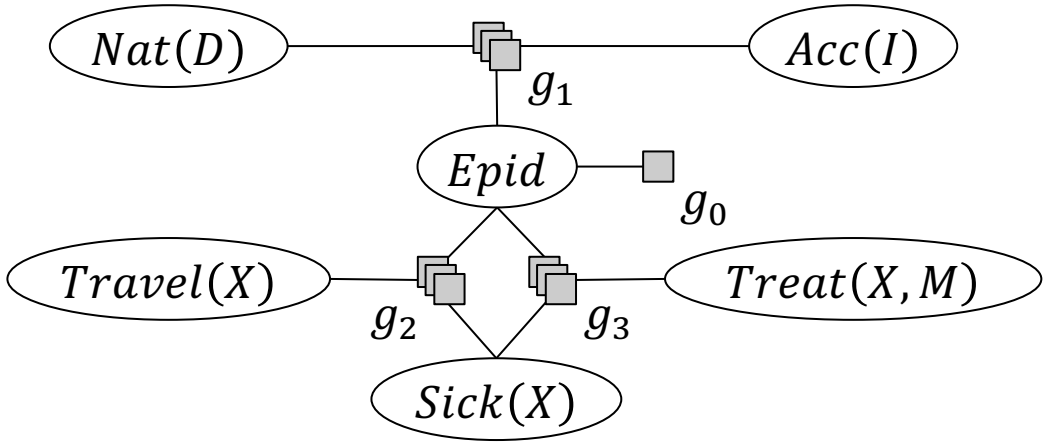
  - Equivalence over semantics:

$$P_G = P_F = P_{\boldsymbol{R}}$$



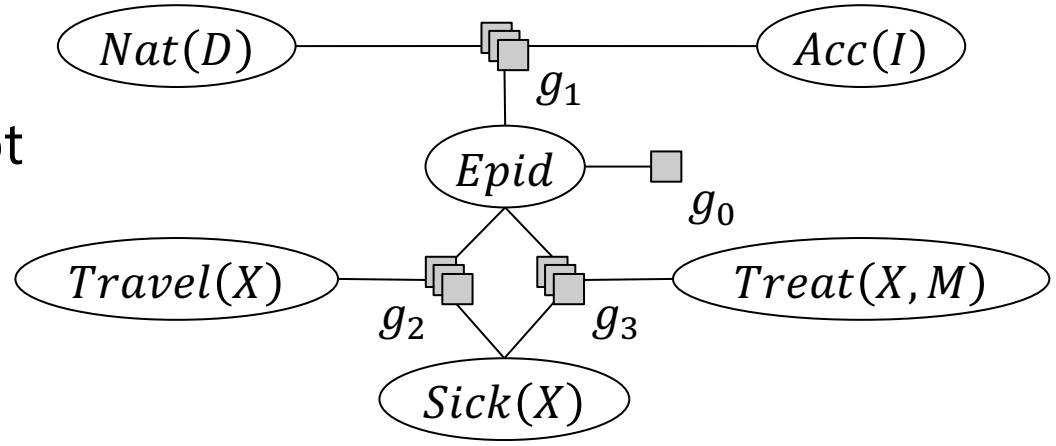
Inference Tasks

- Query Answering Problem (as before)
  - Compute an answer to a query $P(\boldsymbol{S}|\boldsymbol{T})$ given a model G representing the full joint probability distribution P_G
    - Avoid grounding (parts of) G
  - E.g.,
    - $P(Treat(eve, m_1))$
    - $P(Travel(eve), Epid)$
    - $P(Sick(eve)|Epid)$
    - $P(Epid|Sick(eve) = true)$
    - $P(\#_E[Epid(E)])$
    - $P(\#_E[Epid(E)] = [2,2])$
- New syntactical construct of a query: parameterised query $P(\boldsymbol{A}_{|C}|\boldsymbol{T})$
  - Represents $P(gr(\boldsymbol{A}_{|C})|\boldsymbol{T})$
  - E.g., $P(Sick(X)_{|\mathrm{T}})$
    - $P(Sick(eve), Sick(alice), Sick(bob))$



Evidence in Parameterised Models

- Observations for groundings of a PRV
  - Can be
    - One of the range values
    - Not available
  - E.g., for $Sick(X)$ with $dom(X) = \{x_1, \dots, x_n\}$
    - $Sick(x_1) = Sick(x_2) = \cdots = Sick(x_{10}) = true$
    - $Sick(x_{11}) = Sick(x_{12}) = \cdots = Sick(x_{20}) = false$
    - Observations for $Sick(x_{21})$ … $Sick(x_n)$ not available
- Compactly encode evidence with PRVs and parfactors



Evidence Parfactors: Example

- Observations of $Sick(X)$ with range value $true$
  - Parfactor $g_e^T = \phi_e^T\big(Sick(X)\big)_{|(X,C_X)}$
    - $C_X = \{x_1, \dots, x_{10}\}$
    - Or: Use new logical variable Y, $dom(Y) = \{x_1, \dots, x_{10}\}$ $\rightarrow \phi_e^T\big(Sick(Y)\big)_{|\mathsf{T}}$

$Sick(X)$	ϕ_e^T
$false$	0
$true$	1

- Observations of $Sick(X)$ with range value $false$
  - Parfactor $g_e^F = \phi_e^F\big(Sick(X)\big)_{|(X,C_X)}$
    - $C_X = \{x_{11}, \dots, x_{20}\}$
    - Or: Use new logical variable Z, $dom(Z) = \{x_{11}, \dots, x_{20}\}$ $\rightarrow \phi_e^F\big(Sick(Z)\big)_{|\mathsf{T}}$

$Sick(X)$	ϕ_e^F
$false$	1
$true$	0

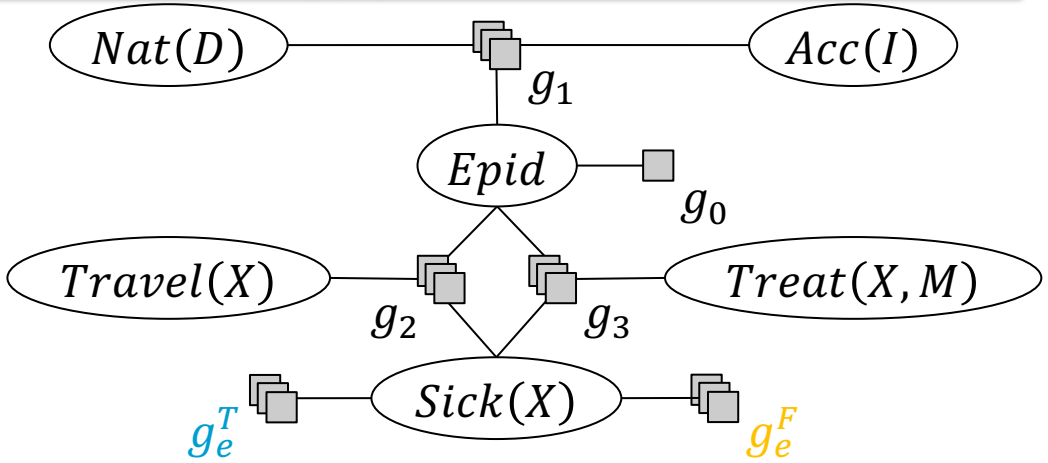
Observations for $Sick(X)$

$Sick(x_1) = Sick(x_2) = \cdots = Sick(x_{10}) = true$

$Sick(x_{11}) = Sick(x_{12}) = \cdots = Sick(x_{20}) = false$

Graphical representation does not depict constraints

- g_2, g_3 have T constraints, g_e^T, g_e^F not
- g_e^T, g_e^F apply to a disjoint subset of constants occurring in g_2, g_3

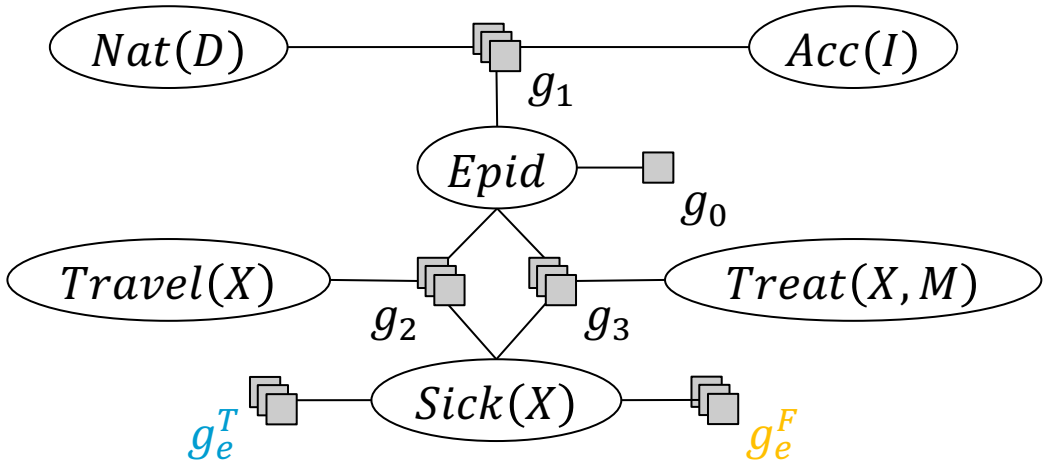


Evidence Parfactors

- Let $\boldsymbol{e}$ be the set of observations (evidence)
- For each PRV A referenced in $rv(\boldsymbol{e})$
  - For each range value r observed for a grounding of A
    - Set up a parfactor g_e with
      - A as argument
      - Range value r mapped to 1 and all range values $r' \neq r$ to 0
      - $C_{\mathcal{X}}$ in constraint $(\mathcal{X}, C_{\mathcal{X}})$ restricting the constants of $\mathcal{X}$ to the constants observed of A with r
        - » Alternative: Use new logical variables $\mathcal{X}'$ in A with domains corresponding to observed constants and T constraints in g_e

Observations for $Sick(X)$

$$Sick(x_1) = Sick(x_2) = \cdots = Sick(x_{10}) = true$$
$$Sick(x_{11}) = Sick(x_{12}) = \cdots = Sick(x_{20}) = false$$



Interim Summary

- First-order view on probabilistic modelling
  - Relations in data
  - Symmetries in graph
- Parameterised models
  - Logical variables with domains of constants
    - Universe
  - Constraints to restrict domains to certain constants
  - PRVs to encode sets of propositional random variables
  - CRVs to encode symmetries within factors
  - Parfactors to build a model with recurring patterns
    - Semantics over grounding and full joint distribution
  - Inference tasks
    - Query terms with logical variables, evidence parfactors

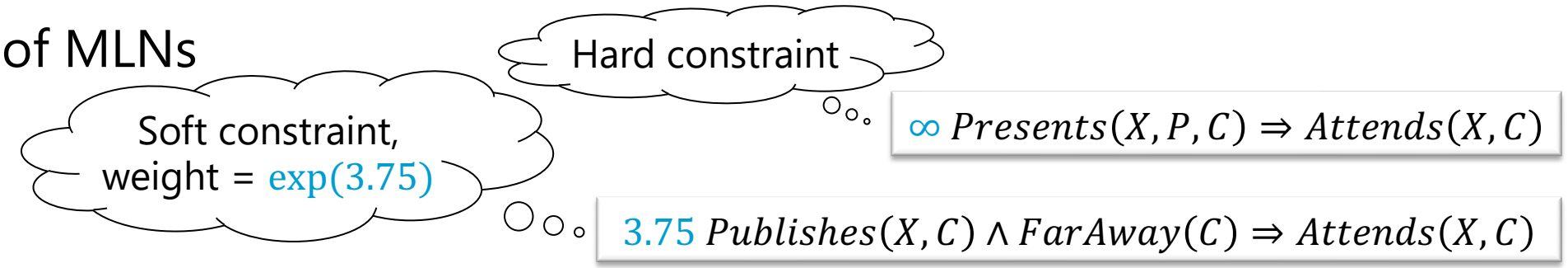
Outline: 2. Probabilistic Relational Models (PRMs)

A. *Parfactor models (PMs)*
   - Motivation: Symmetries and relations
   - Syntax, semantics
   - Graphical representation
   - Inference tasks

B. *Markov logic networks (MLNs)*
   - Syntax, semantics of MLNs
   - Graphical representation
   - Turning MLNs into PMs and vice versa

Markov Logic Networks (MLNs)

- Again: First-order view on probabilistic modelling
- Use logical formulas to specify potential functions
  - Weights for each formula induce a full joint again
    - Lower weight if constraint does not hold all the time
    - ∞ weight for hard constraints, in practice either a high weight or only worlds considered that satisfy hard constraints
- Next slides
  - Syntax of MLNs
  - Semantics of MLNs



Matthew Richardson and Pedro Domingos: Markov Logic Networks. In: *Machine Learning*, 2006.

Markov Logic Networks (MLNs)

- Weighted formulas for modelling constraints (≠ domain constraints)
- An MLN is a set of constraints $\Psi = \{(w_i, \psi_i)\}_{i=1}^{n}$
  - $w_i \in \mathbb{R}^+$ weight
  - ψ_i FOL formula
    - Originally without a constraint language for domains
    - Mostly used with implicit all-quantifiers
  - Implicitly connected via conjunction
    - I.e., set of formulas ψ_i = knowledge base/theory

Hard constraint

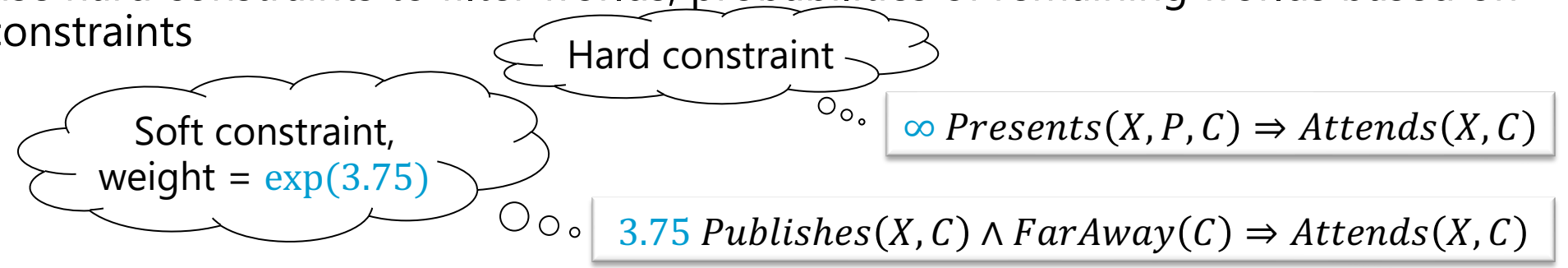
$\infty\ Presents(X, P, C) \Rightarrow Attends(X, C)$

Soft constraint, weight = $\exp(3.75)$

$3.75\ Publishes(X, C) \wedge FarAway(C) \Rightarrow Attends(X, C)$

Intuition behind Weights

- Soften logic using weights
  - Worlds that violate constraint become less likely but not impossible
    - As w_i increases, so does the strength of ψ_i
    - Infinite weight: Hard constraint = pure logic formula
      - Probabilities of worlds that do not satisfy hard constraint set to 0
    - Standard MLNs only use weights in $\mathbb{R}$; otherwise semantics break (special handling of hard constraints required)
      - E.g., use hard constraints to filter worlds, probabilities of remaining worlds based on soft constraints



Grounding

- Each (w_i, ψ_i) represents a set of *propositional* sentences, each sentence with weight w_i
  - One sentence for each possible substitution of the free variables $free(\psi_i)$ in ψ_i given a finite domain (or a constraint set) D over $free(\psi_i)$
    - $\theta_D = \mathrm{U}_{\boldsymbol{d} \in D}\{\mathrm{U}_{t \in \boldsymbol{d}}\{X_d \rightarrow t\}\}$
  - Example: MLN $\Psi = \{(w_i, \psi_i)\}_{i=1}^{2}$
    - Domains
      - $dom(X) = \{alice\}$
      - $dom(P) = \{p_1, p_2\}$
      - $dom(C) = \{ijcai, ki\}$
    - Groundings on the right

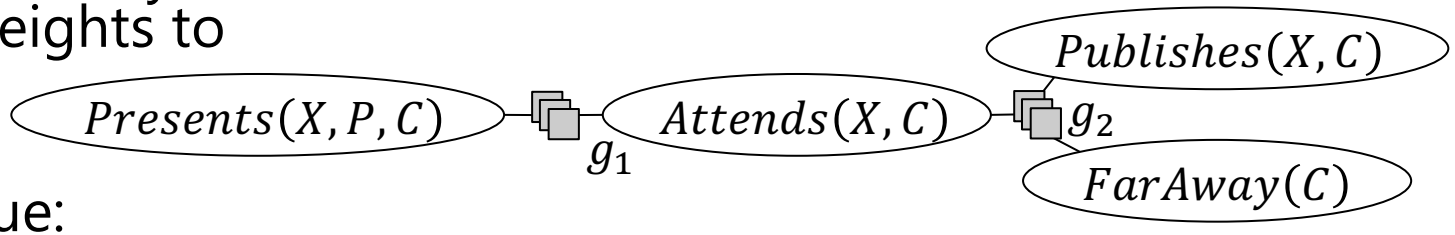
$$(10, Presents(alice, p_1, ijcai) \Rightarrow Attends(alice, ijcai))$$
$$(10, Presents(alice, p_1, ki) \Rightarrow Attends(alice, ki))$$
$$(10, Presents(alice, p_2, ijcai) \Rightarrow Attends(alice, ijcai))$$
$$(10, Presents(alice, p_2, ki) \Rightarrow Attends(alice, ki))$$
$$(3.75, Publishes(alice, ijcai) \wedge FarAway(ijcai) \Rightarrow Attends(alice, ijcai))$$
$$(3.75, Publishes(alice, ki) \wedge FarAway(ki) \Rightarrow Attends(alice, ki))$$

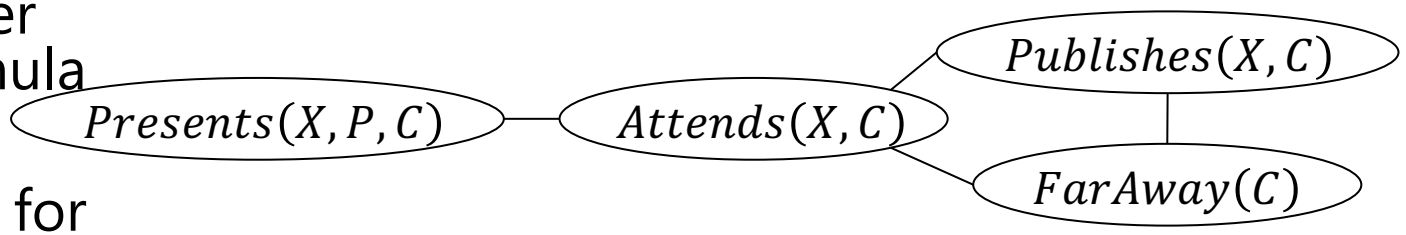
$$10\ Presents(X, P, C) \Rightarrow Attends(X, C)$$

$$3.75\ Publishes(X, C) \wedge FarAway(C) \Rightarrow Attends(X, C)$$

MLN: Graphical Representation?

- Usually not depicted by a graph but by the logical formulas with their weights to the left
- Since the name invokes Markov networks, let us build an analogue:
  - Logical atoms as nodes
  - Edges between atoms whenever atoms occur together in a formula
    - Form cliques in graph
    - Potential functions (factors) for cliques from weights
    - Can also build an equivalent parfactor graph as graphical representation



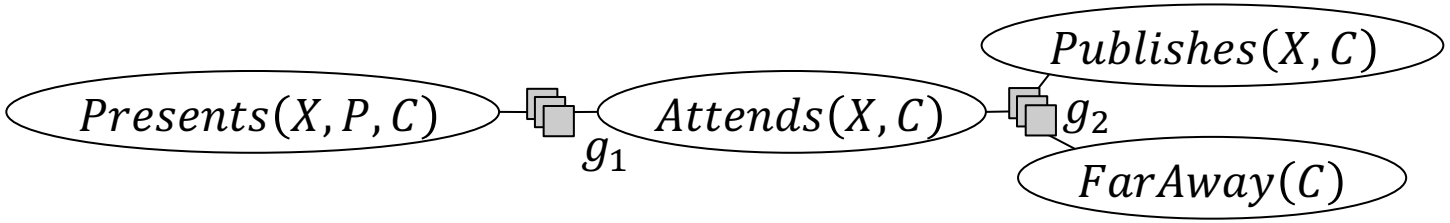


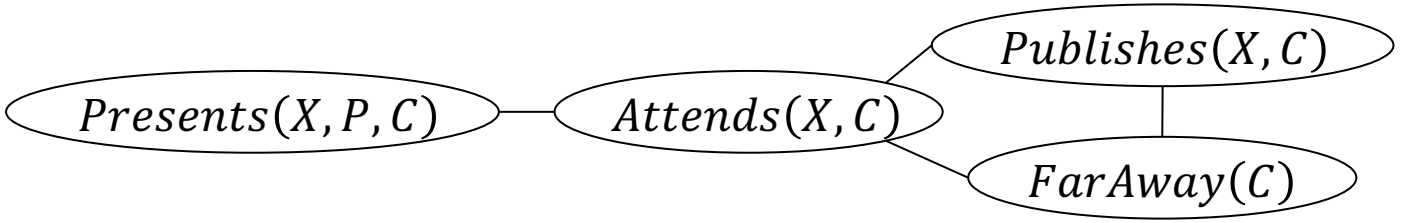
$10\ Presents(X, P, C) \Rightarrow Attends(X, C)$

$3.75\ Publishes(X, C) \wedge FarAway(C) \Rightarrow Attends(X, C)$

Weights to Potential Functions (Factors)

- How to get from weights to potential functions (factors)?
  - Arguments = Atoms
    - Only Boolean ranges
  - Map input to $\exp(w_i)$ if input makes ψ_i true
    - Input = assignments to atoms in ψ_i
  - Otherwise map to $\exp(0) = 1$
  - If indeed $w_i = \infty$: choose large number
    - In implementation: maximum number of chosen number format



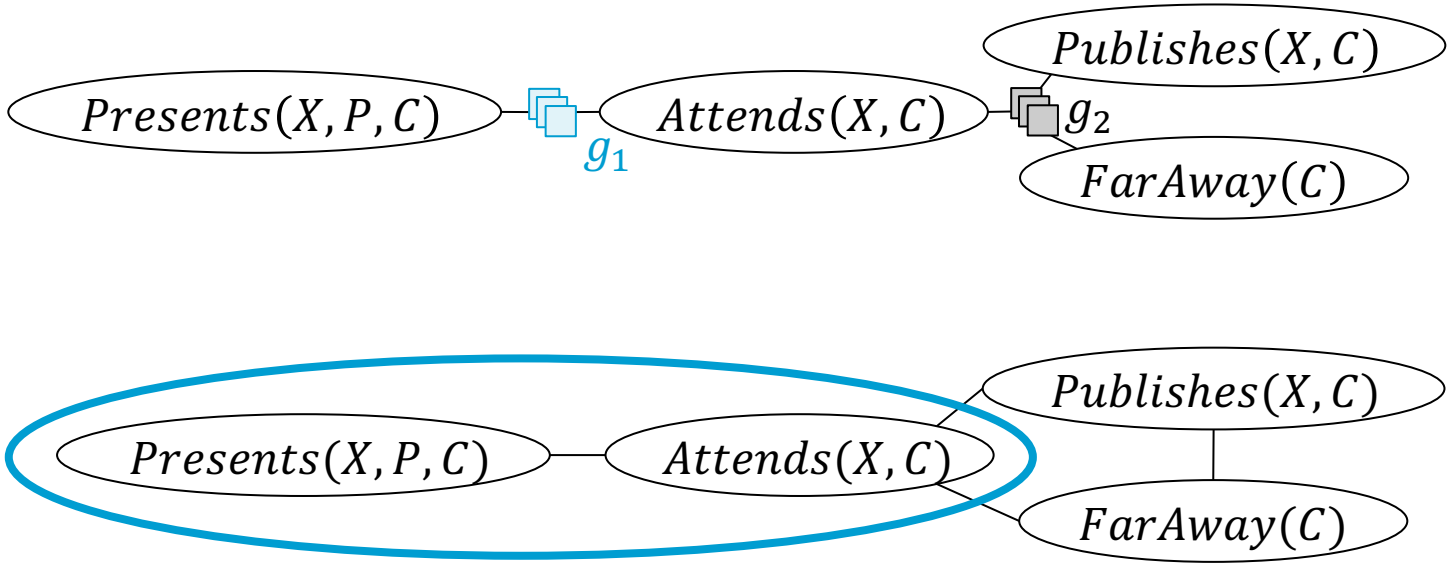


$10\ Presents(X, P, C) \Rightarrow Attends(X, C)$

$3.75\ Publishes(X, C) \wedge FarAway(C) \Rightarrow Attends(X, C)$

Weights to Potential Functions (Factors) – Examples

$Presents(X,P,C)$	$Attends(X,C)$	ϕ_1
$false$	$false$	exp 10
$false$	$true$	exp 10
$true$	$false$	exp 0
$true$	$true$	exp 10

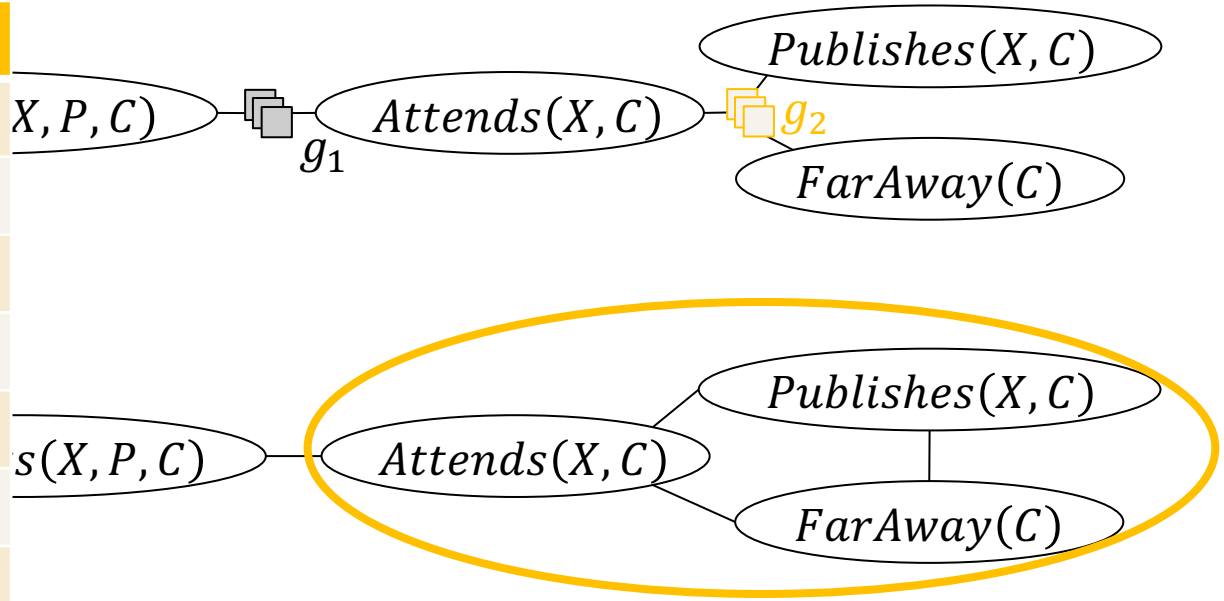


$10\ Presents(X,P,C) \Rightarrow Attends(X,C)$

$3.75\ Publishes(X,C) \wedge FarAway(C) \Rightarrow Attends(X,C)$

Weights to Potential Functions (Factors) – Examples

$Publishes(X,C)$	$FarAway(C)$	$Attends(X,C)$	ϕ
$false$	$false$	$false$	exp 3.75
$false$	$false$	$true$	exp 3.75
$false$	$true$	$false$	exp 3.75
$false$	$true$	$true$	exp 3.75
$true$	$false$	$false$	exp 3.75
$true$	$false$	$true$	exp 3.75
$true$	$true$	$false$	exp 0
$true$	$true$	$true$	exp 3.75



10 $Presents(X,P,C) \Rightarrow Attends(X,C)$

3.75 $Publishes(X,C) \wedge FarAway(C) \Rightarrow Attends(X,C)$

Graphical Representation for Groundings

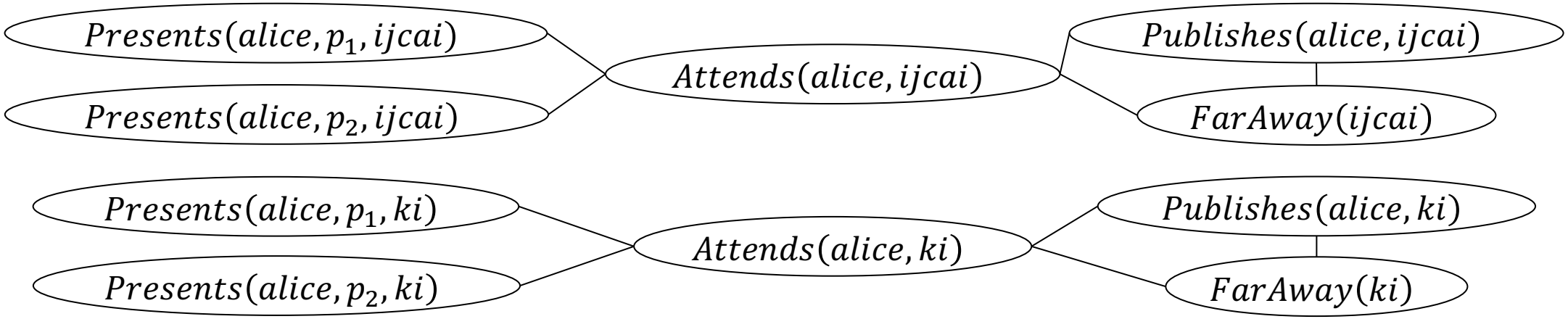
- Each (w_i, ψ_i) represents a set of propositional sentences, each sentence with weight w_i
- Can also form a Markov network graphical representation (or factor graph) for this set of propositional sentences
  - Nodes = random variables = ground atoms
  - Edges connect literals that appear together in a sentence
  - Potential functions (as before, now with instances)
    - Map to $\exp(w_i)$ if assignment to ground atoms makes ψ_i true
    - Otherwise map to $\exp(0) = 1$
    - If indeed $w_i = \infty$: choose large number

$$10\ Presents(X, P, C) \Rightarrow Attends(X, C)$$

$$3.75\ Publishes(X, C) \wedge FarAway(C) \Rightarrow Attends(X, C)$$

Instances: Example – MN

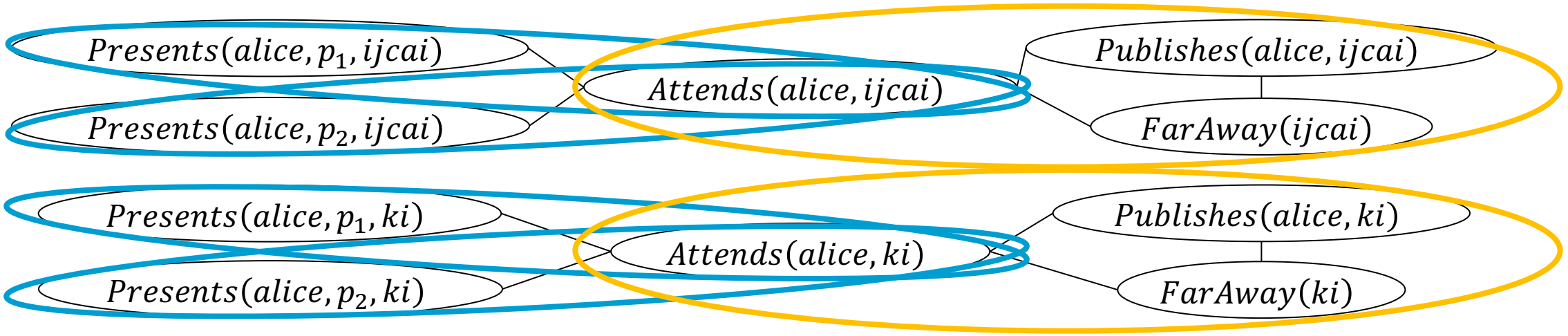
- $\left(10, Presents(alice, p_1, ijcai) \Rightarrow Attends(alice, ijcai)\right)$
- $\left(10, Presents(alice, p_1, ki) \Rightarrow Attends(alice, ki)\right)$
- $\left(10, Presents(alice, p_2, ijcai) \Rightarrow Attends(alice, ijcai)\right)$
- $\left(10, Presents(alice, p_2, ki) \Rightarrow Attends(alice, ki)\right)$
- $\left(3.75, Publishes(alice, ijcai) \wedge FarAway(ijcai) \Rightarrow Attends(alice, ijcai)\right)$
- $\left(3.75, Publishes(alice, ki) \wedge FarAway(ki) \Rightarrow Attends(alice, ki)\right)$



Example – Potential Functions

$Presents(x,p,c)$	$Attends(x,c)$	ϕ_1
$false$	$false$	exp 10
$false$	$true$	exp 10
$true$	$false$	exp 0
$true$	$true$	exp 10

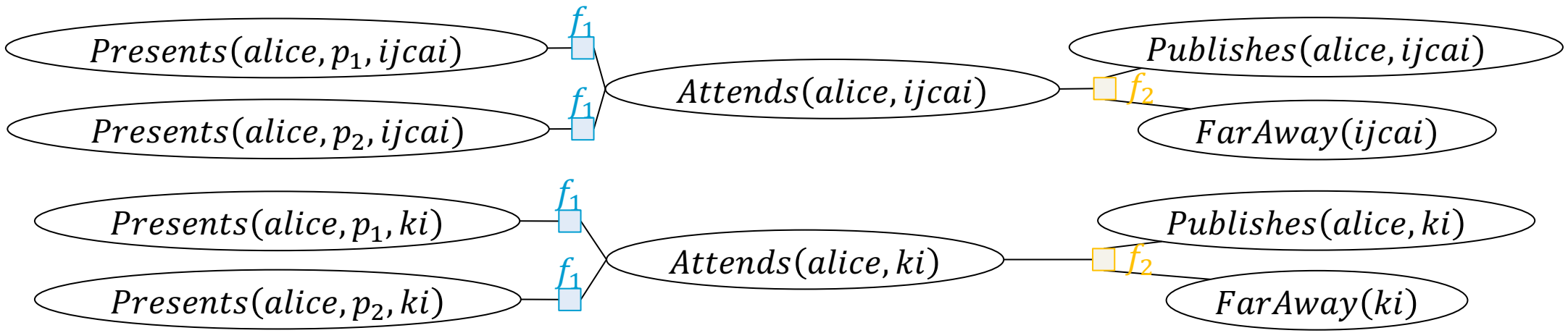
$Publishes(x,c)$	$FarAway(c)$	$Attends(x,c)$	ϕ_2
$false$	$false$	$false$	exp 3.75
$false$	$false$	$true$	exp 3.75
$false$	$true$	$false$	exp 3.75
$false$	$true$	$true$	exp 3.75
$true$	$false$	$false$	exp 3.75
$true$	$false$	$true$	exp 3.75
$true$	$true$	$false$	exp 0
$true$	$true$	$true$	exp 3.75



Example – Factor Graph

$Presents(x,p,c)$	$Attends(x,c)$	ϕ_1
$false$	$false$	exp 10
$false$	$true$	exp 10
$true$	$false$	exp 0
$true$	$true$	exp 10

$Publishes(x,c)$	$FarAway(c)$	$Attends(x,c)$	ϕ_2
$false$	$false$	$false$	exp 3.75
$false$	$false$	$true$	exp 3.75
$false$	$true$	$false$	exp 3.75
$false$	$true$	$true$	exp 3.75
$true$	$false$	$false$	exp 3.75
$true$	$false$	$true$	exp 3.75
$true$	$true$	$false$	exp 0
$true$	$true$	$true$	exp 3.75



MLNs: Semantics

- MLN $\Psi = \{(w_i, \psi_i)\}_{i=1}^{n}$, with $w_i \in \mathbb{R}$, induces a probability distribution over possible interpretations ω (world) of the grounded atoms in Ψ

$$\omega \in \{true, false\}^N$$

  - N = the number of ground atoms in the grounded Ψ
  - Probability of one interpretation ω

$$P(\omega) = \frac{1}{Z} \exp\left(\sum_{i=1}^{n} w_i n_i(\omega)\right)$$

    - $n_i(\omega)$ = number of propositional sentences of ψ_i that evaluate to $true$ given the assignments of ω

MLNs: Semantics – Derivation

- Let an MLN $\Psi = \{(w_i, \psi_i)\}_{i=1}^{n}$ and a domain D be given
- The grounded MLN is the following

$$gr(\Psi) = \bigcup_{i=1}^{n} \bigcup_{\theta \in \theta_D} \{(w_i, \psi_i \theta)\}$$

- The semantics of $gr(\Psi)$ induces a probability distribution over possible interpretations ω
  - ω assigns a truth value to each (ground) atom in $gr(\Psi)$
  - I.e., a normalised product over all weights of formulas (potential functions of cliques) with $\exp w_j$ if ω makes $\psi_j \theta$ true and $\exp 0$ otherwise, i.e.,

$$P(\omega) = \frac{1}{\mathrm{Z}} \prod_{j=1}^{|gr(\Psi)|} \exp(w_j) = \frac{1}{Z} weight(\omega)$$

    - Product called weight of a world

MLNs: Semantics – Derivation

- Consider the weight of ω

$$weight(\omega) = \prod_{j=1}^{|gr(\Psi)|} \exp(w_j)$$

- To simplify, use the following:
  - Sentences in $gr(\Psi)$ are groundings of n formulas where each set of propositional sentences per formula carries the same weight
  - If a propositional sentences evaluates to $false$, its contribution to the product is $\exp(0) = 1$.

→ Rewrite expression:

$$weight(\omega) = \prod_{i=1}^{n} \exp(w_i)^{n_i(\omega)} = \prod_{i=1}^{n} \exp\left(w_i \cdot n_i(\omega)\right)$$

  - $n_i(\omega)$ = number of propositional sentences of ψ_i that evaluate to $true$ given ω

MLNs: Semantics – Derivation

- Therefore, $\Psi = \{(w_i, \psi_i)\}_{i=1}^{n}$ induces a probability distribution over possible worlds ω

$$P(\omega) = \frac{1}{Z} weight(\omega)$$
$$= \frac{1}{Z} \prod_{i=1}^{n} \exp\left(w_i \cdot n_i(\omega)\right)$$
$$= \frac{1}{Z} \exp\left(\sum_{i=1}^{n} w_i n_i(\omega)\right)$$

  - $Z = \sum_{\omega \in \{true, false\}^N} weight(\omega)$
    - N = the number of ground atoms in $gr(\Psi)$

Why exp?

- Weight of a world ω:

$$weight(\omega) = \exp\left(\sum_{i=1}^{n} w_i n_i(\omega)\right)$$

- Taking the logarithm yields

$$lweight(\omega) = \ln \exp\left(\sum_{i=1}^{n} w_i n_i(\omega)\right) = \sum_{i=1}^{n} w_i n_i(\omega)$$

  - Sum allows for component-wise optimisation during weight learning
  - Referred to as log-linear models
- Semantics:

$$P(\omega) = \frac{1}{Z'} \sum_{i=1}^{n} w_i n_i(\omega)$$

  - $Z' = \sum_{\omega \in \{true,false\}^N} lweight(\omega) = \sum_{\omega \in \{true,false\}^N} \sum_{i=1}^{n} w_i n_i(\omega)$

Transforming MLNs into PMs

- Follows the same idea of generating a graphical representation
  - Logical atoms = PRVs $\mathcal{A}$
  - Potential function
    - Map a to $\exp(w_i)$ if truth values in a as assignments to atoms makes ψ_i true
    - Otherwise map to $\exp(0) = 1$
    - If indeed $w_i = \infty$: choose large number
      - In implementation: maximum number of chosen number encoding
  - Constraints
    - If no domain constraints given: $(\mathcal{X}, C_{\mathcal{X}}) = \top$, i.e.,
      $$\mathcal{X} = lv(\mathcal{A}), C_{\mathcal{X}} = \times_{X \in \mathcal{X}} dom(X)$$
    - Otherwise: build constraint according to domain constraints

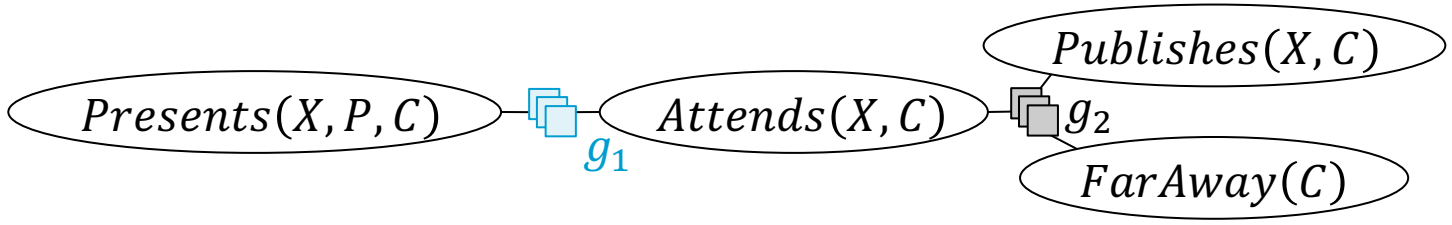
10 $Presents(X, P, C) \Rightarrow Attends(X, C)$

3.75 $Publishes(X, C) \wedge FarAway(C) \Rightarrow Attends(X, C)$

Transforming MLNs into PMs: Example

- E.g.,
  - $g_1 = \phi\big(Presents(X,P,C), Attends(X,C)\big)_{|C_1}$
    - $C_1 = \big((X,P,C), dom(X) \times dom(P) \times dom(C)\big)$

$Presents(X,P,C)$	$Attends(X,C)$	ϕ_1
$false$	$false$	exp 10
$false$	$true$	exp 10
$true$	$false$	exp 0
$true$	$true$	exp 10



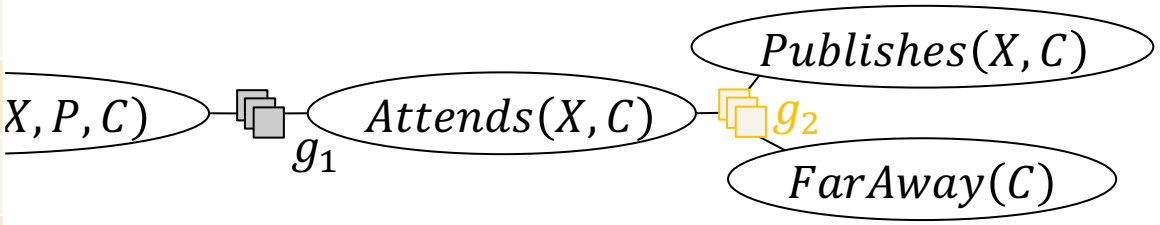
$10\ Presents(X,P,C) \Rightarrow Attends(X,C)$

$3.75\ Publishes(X,C) \wedge FarAway(C) \Rightarrow Attends(X,C)$

Transforming MLNs into PMs: Example

- $g_2 = \phi\big(Publishes(X,C), FarAway(C), Attends(X,C)\big)_{|C_2}$
  - $C_2 = \big((X,C), dom(X) \times dom\mathcal{D}(C)\big)$

$Publishes(X,C)$	$FarAway(C)$	$Attends(X,C)$	ϕ
$false$	$false$	$false$	exp 3.75
$false$	$false$	$true$	exp 3.75
$false$	$true$	$false$	exp 3.75
$false$	$true$	$true$	exp 3.75
$true$	$false$	$false$	exp 3.75
$true$	$false$	$true$	exp 3.75
$true$	$true$	$false$	exp 0
$true$	$true$	$true$	exp 3.75



10 $Presents(X,P,C) \Rightarrow Attends(X,C)$

3.75 $Publishes(X,C) \wedge FarAway(C) \Rightarrow Attends(X,C)$

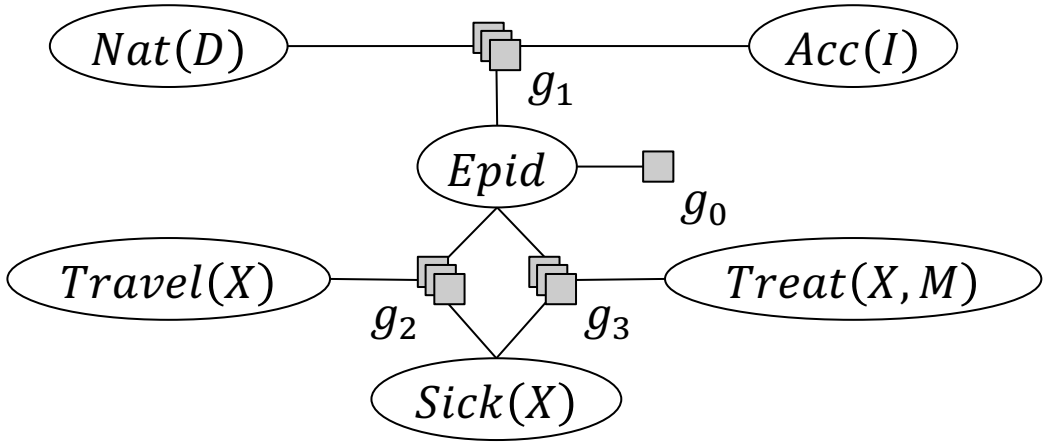
Transforming PMs into MLNs

- Inverse of Transforming MLNs into PMs
- Given $G = \{\phi_i(\mathcal{A}_i)_{|C_i}\}_{i=1}^{n}$
  - Precondition: All $A \in rv(G)$ need to have Boolean ranges
  - PRVs $\mathcal{A}$ = logical atoms
  - Constraint C_i
    - If $C_i = \top$: original MLN without further restrictions
    - Otherwise: encode in constraint set per formula
- Potential function ϕ_i
  - For each input-output mapping $a \to p$
    - Build formula $\psi = \bigwedge_{A \in \mathcal{A}} t(A, a)$ with weight $w = \ln p$
      - $t(A, a) = \begin{cases} a & \text{Assignment of } A \text{ in } a \text{ is } true \\ \neg a & \text{Assignment of } A \text{ in } a \text{ is } false \end{cases}$
      - If the potentials are in $[0,1]$, then w is negative
        - » If potential is 0, choose large negative number
          - » Compare to the inverse with a weight of infinity
      - Best: ensure that potentials are in either $(0,1]$ or $\mathbb{R}_{\geq 1}$

Inverse Example

- E.g., $G = \{g_i\}_{i=0}^{3}$
  - $g_1 = \phi(Epid)$
  - $g_1 = \phi\big(Epid, Nat(D), Acc(I)\big)$
  - $g_2 = \phi\big(Travel(X), Epid, Sick(X)\big)$
  - $g_3 = \phi\big(Epid, Sick(X), Treat(X, M)\big)$
  - Logical atoms:

    $Epid, Nat(D), Acc(I),$
    $Travel(X), Sick(X),$
    $Treat(X, M)$
  - Domains as given for G



Inverse Example

- Weighted formulas (w_i, ψ_i) for g_2
  - $(\ln 5, \neg travel(X) \wedge \neg epid \wedge \neg sick(X))$
  - $(\ln 1, \neg travel(X) \wedge \neg epid \wedge sick(X))$
  - $(\ln 4, \neg travel(X) \wedge epid \wedge \neg sick(X))$
  - $(\ln 6, \neg travel(X) \wedge epid \wedge sick(X))$
  - $(\ln 4, travel(X) \wedge \neg epid \wedge \neg sick(X))$
  - $(\ln 6, travel(X) \wedge \neg epid \wedge sick(X))$
  - $(\ln 2, travel(X) \wedge epid \wedge \neg sick(X))$
  - $(\ln 9, travel(X) \wedge epid \wedge sick(X))$
- Same has to be done for g_0, g_1, g_3

$Travel(X)$	$Epid$	$Sick(X)$	ϕ_2
$false$	$false$	$false$	5
$false$	$false$	$true$	1
$false$	$true$	$false$	4
$false$	$true$	$true$	6
$true$	$false$	$false$	4
$true$	$false$	$true$	6
$true$	$true$	$false$	2
$true$	$true$	$true$	9

If some potentials appear multiple times, use an algorithm such as Quine-McCluskey to minimise formulas needed to encode the same information [Marwitz et al. (2021)]

Florian Marwitz, Tanya B, and Ralf Möller: A First Step Towards Even More Sparse Encodings of Probability Distributions. In *ILP-21 Proceedings of the 30th International Conference on Inductive Logic Programming*, 2021.

Inverse Example

- Consider $\phi\big(Travel(X), Epid, Sick(X)\big)$ as given by the table on the right
- Only two weighted formulas (w_i, ψ_i) necessary
  - $\big(\ln 2\,, \neg travel(X) \vee \neg epid \vee \neg sick(X)\big)$
  - $\big(\ln 7\,, travel(X) \wedge epid \wedge sick(X)\big)$
  - If potential of 1 instead of 2, would reduce to
    - $\big(\ln 7\,, travel(X) \wedge epid \wedge sick(X)\big)$
    - Assignments that do not make the formula true automatically get weight of $0 = \ln 1$

$Travel(X)$	$Epid$	$Sick(X)$	ϕ
$false$	$false$	$false$	2
$false$	$false$	$true$	2
$false$	$true$	$false$	2
$false$	$true$	$true$	2
$true$	$false$	$false$	2
$true$	$false$	$true$	2
$true$	$true$	$false$	2
$true$	$true$	$true$	7

Size of resulting MLN depends on the *local symmetries* in the potentials!

Florian Marwitz, Tanya B, and Ralf Möller: A First Step Towards Even More Sparse Encodings of Probability Distributions. In *ILP-21 Proceedings of the 30th International Conference on Inductive Logic Programming*, 2021.

MLNs vs. PMs

- MLNs: soft constraints in FOL-DC
  - Requires knowing interactions to build formulas with logical connectives
  - Allows humans to interpret a knowledge base more easily
  - Space-efficient encoding *if* different truth value assignments have the same weight *and* fewer rules can encode them
- PMs are more blurred
  - Interaction described via distributions
    - Does not enforce connectives between variables
    - But: Interpretation not that easy
- In propositional case
  - MNs and FGs are different graphical representations of the same factor-based model
- In relational case
  - MLNs and PMs are more different
    - Semantics still the same: full joint
    - Can transform one into the other → same expressivity
    - But the syntax is considered differently
- Possible to start with PMs
  - Extract rules if necessary

Interim Summary

- MLNs
  - Weighted formulas
  - Semantics again over groundings and full joint distribution
  - Log-linear version allows for local optimisation during learning
- Transformations between MLNs and PMs
  - One weighted rule → one potential function
  - One potential function → possibly # of input-output mappings different rules
  - If local symmetries exist (in potentials or weights), then MLNs offer compact encoding

Other Formalisms*

- Relational BNs
  - Nodes in BNs are n-ary relations, labelled with so-called probability formulas, which define the semantics as a probability measure over interpretations of the relations
  - Implementation: http://people.cs.aau.dk/~jaeger/Primula/
- Hinge-loss MNs
  - Continuous variables in the $[0,1]$ unit interval combined in constraints of first-order logic syntax
    - Distance of each constraint to satisfaction = hinge loss
  - Implemented with Probabilistic Soft Logic https://psl.linqs.org
- ProbLog: Probabilistic Prolog (probabilistic logic programming)
  - Defines a probability distribution over logic programs
  - Typically Horn clauses annotated with probabilities
  - Implementation: https://dtai.cs.kuleuven.be/problog/

* Relations not necessarily used for efficient inference (point of the next lectures)

Manfred Jaeger: Relational Bayesian Networks. In: *UAI-97 Proceedings of the 13th Conference on Uncertainty in Artificial Intelligence*, 1997.
Stephen H. Bach, Matthias Broecheler, Bert Huang, and Lise Getoor: Hinge-loss Markov Random Fields and Probabilistic Soft Logic. In: *Journal of Machine Learning Research*, 2017
Luc De Raedt, Angelika Kimmig, and Hannu Toivonen: ProbLog: A Probabilistic Prolog and its Application in Link Discovery. In: IJCAI-07 Proceedings of the 20th International Joint Conference on Artificial Intelligence, 2007.

Outline: 2. Probabilistic Relational Models (PRMs)

A. *Parfactor models (PMs)*
   - Motivation: Symmetries and relations
   - Syntax, semantics
   - Graphical representation
   - Inference tasks

B. *Markov logic networks (MLNs)*
   - Syntax, semantics of MLNs
   - Graphical representation
   - Turning MLNs into PMs and vice versa

→ Lifted Inference