Lecture

Quantum Computing

(CS5070)

Quantum Error Correction

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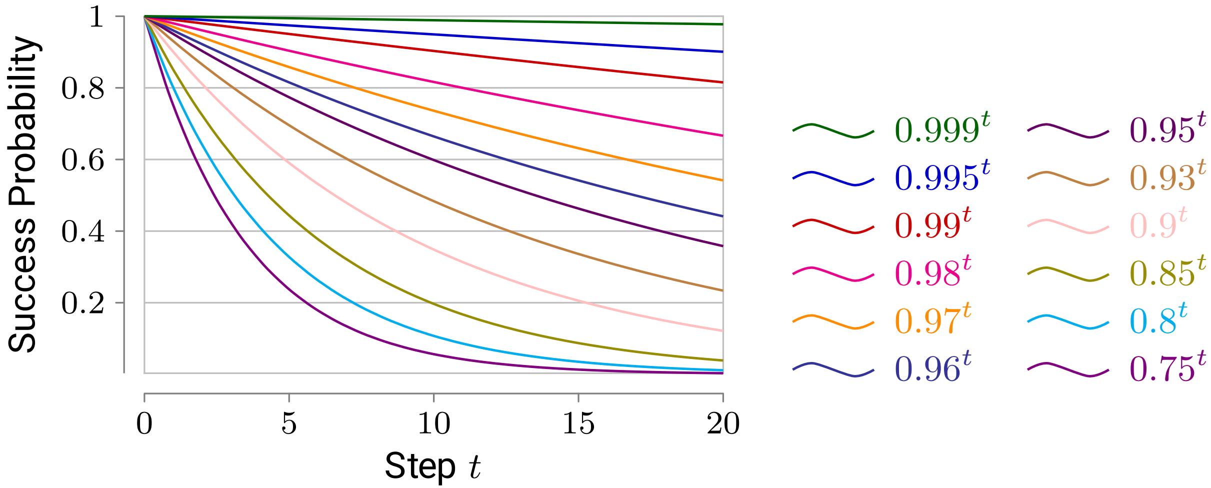
https://www.ifis.uni-luebeck.de/index.php?id=groppe

Error Correction

- Physical devices are imperfect
  - for quantum devices even more true than for classical devices
- Interactions with the environment
  - makes errors (like decoherence for quantum devices) more likely
- Error must be controlled or compensated
  - Probability of one step to succeed: p
  - Probability of t steps to succeed: p^t



Quantum Error

- A quantum error can occur at any time
  - changing the superposition $\alpha|0\rangle + \beta|1\rangle$ of a qubit to $\alpha'|0\rangle + \beta'|1\rangle$
  - quantum errors can occur in several qubits at the same time
- Examples of single-qubit errors:

Name	Operator	Effect on $|0\rangle$	Effect on $|1\rangle$
Bit Flip	X	$X|0\rangle = |1\rangle$	$X|1\rangle = |0\rangle$
Phase Flip	Z	$Z|0\rangle = |0\rangle$	$Z|1\rangle = -|1\rangle$
Rotation	R_θ	$R_\theta|0\rangle = |0\rangle$	$R_\theta|1\rangle = e^{i\theta}|1\rangle$
(Full) Decoherence	$\alpha|0\rangle + \beta|1\rangle \to \{|0\rangle, |1\rangle\}$		

Decoherence

- [SAG'19] runs amplitude and phase damping simulations
  - Decoherence (of amplitude and phase) depends on
    - type of gate
    - input
    - quantum circuit depth
  - simple (logical) optimizations resulting in lower-depth circuits can improve the decoherence by 20%
    - comparing full adders with depths 6 and 9

Input	Gates/Circuits from lowest to highest (amplitude) decoherence
$\frac{\|000\rangle+\|001\rangle+\|010\rangle+\|011\rangle+\|100\rangle+\|101\rangle+\|110\rangle+\|111\rangle}{\sqrt{2^3}}$	Phase, T, Toffoli, Fredkin, CNOT, H (relatively close together)
$\|110\rangle$	CNOT, H, *Others*, Toffoli
$\|101\rangle$	H, *Others*, CNOT
Any input	Full adders (3 qubits $\longrightarrow$ sum+carry): QCKT-1 (depth 6), QCKT-2 (depth 9)

(Simple) Classical Repetition Code

- Classical data:
  - Repetition code for correction of a single bit-flip:
    - $0 \rightarrow 000$
    - $1 \rightarrow 111$
  - A single bit flip error
    - Choosing the majority of the three bits, e.g. $010 \rightarrow 0$
  - In case of rare errors: 1 error is more likely than 2

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- How to adapt this approach for quantum error correction?

Barriers to Quantum Error Correction

- Measurement destroys superpositions
- No-cloning theorem prevents repetition
- Correction of multiple types of errors
  - e.g., bit flip and phase errors
- How to correct continuous errors and decoherence?

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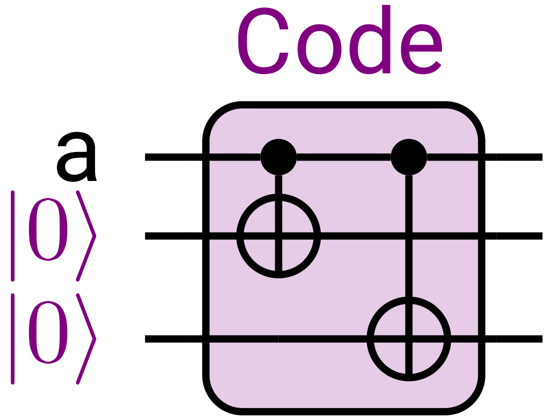
- Higher circuit depths due to quantum error correction approaches
  - Can state-of-the-art quantum computers process these circuit depths?

Quantum Repetition Code correcting bit flip errors

- How to construct quantum circuit for quantum repetition code?
  - $|0\rangle \rightarrow |000\rangle$
  - $|1\rangle \rightarrow |111\rangle$
  - $\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$

Quantum Repetition Code - Generate Code

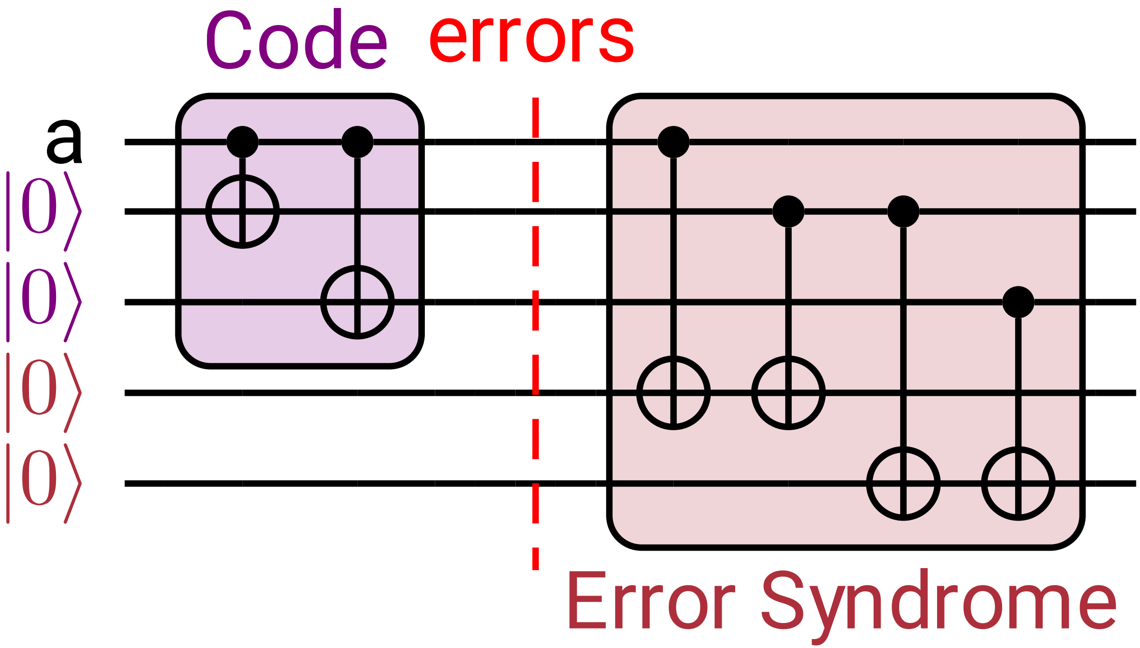


$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$

Next step:

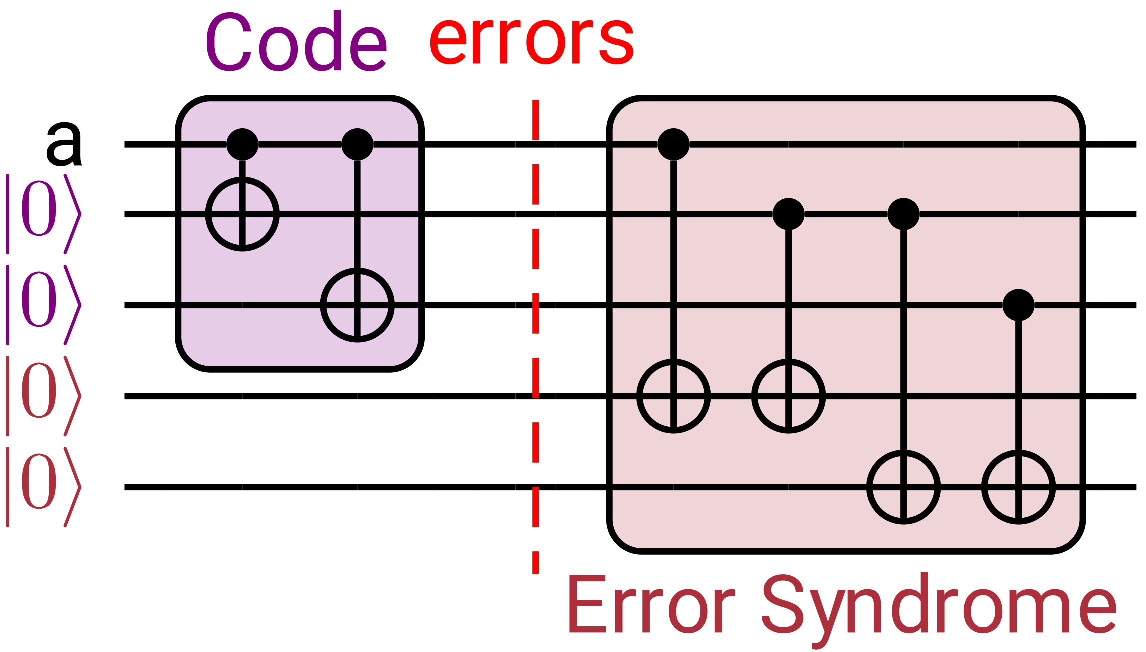
How to detect qubit flips without measurement and cloning?

Quantum Repetition Code - Error Syndrome



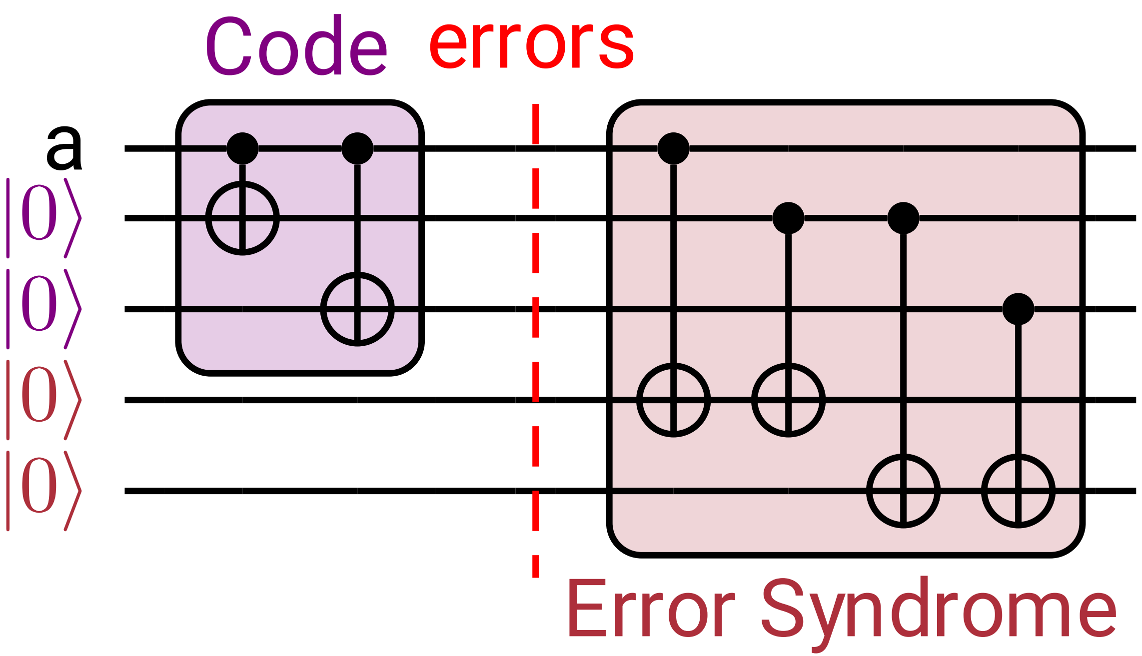
Input	Error Syn-drome	Action
$\|000\rangle$		
$\|111\rangle$		
$\|010\rangle$		
$\|101\rangle$		
$\|100\rangle$		
$\|011\rangle$		
$\|001\rangle$		
$\|110\rangle$		

Quantum Repetition Code - Error Syndrome



Input	Error Syn-drome	Action
$\lvert 000\rangle$ $\lvert 111\rangle$	$\lvert 00\rangle$	None
$\lvert 010\rangle$ $\lvert 101\rangle$	$\lvert 11\rangle$	Flip 2nd qubit
$\lvert 100\rangle$ $\lvert 011\rangle$	$\lvert 10\rangle$	Flip 1st qubit
$\lvert 001\rangle$ $\lvert 110\rangle$	$\lvert 01\rangle$	Flip 3rd qubit

Quantum Repetition Code - Error Syndrome



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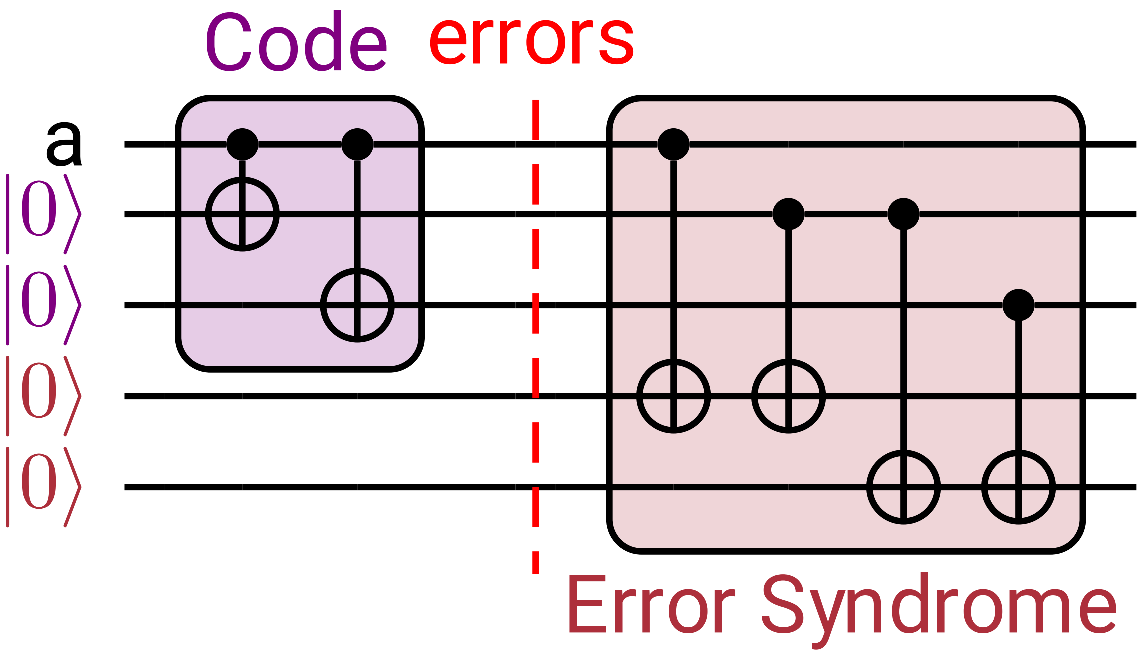
Incomplete bit flip:

Input: $A = |0\rangle \otimes (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) \otimes |0\rangle = \alpha \cdot |000\rangle + \beta \cdot |010\rangle$

Quantum Repetition Code - Error Syndrome



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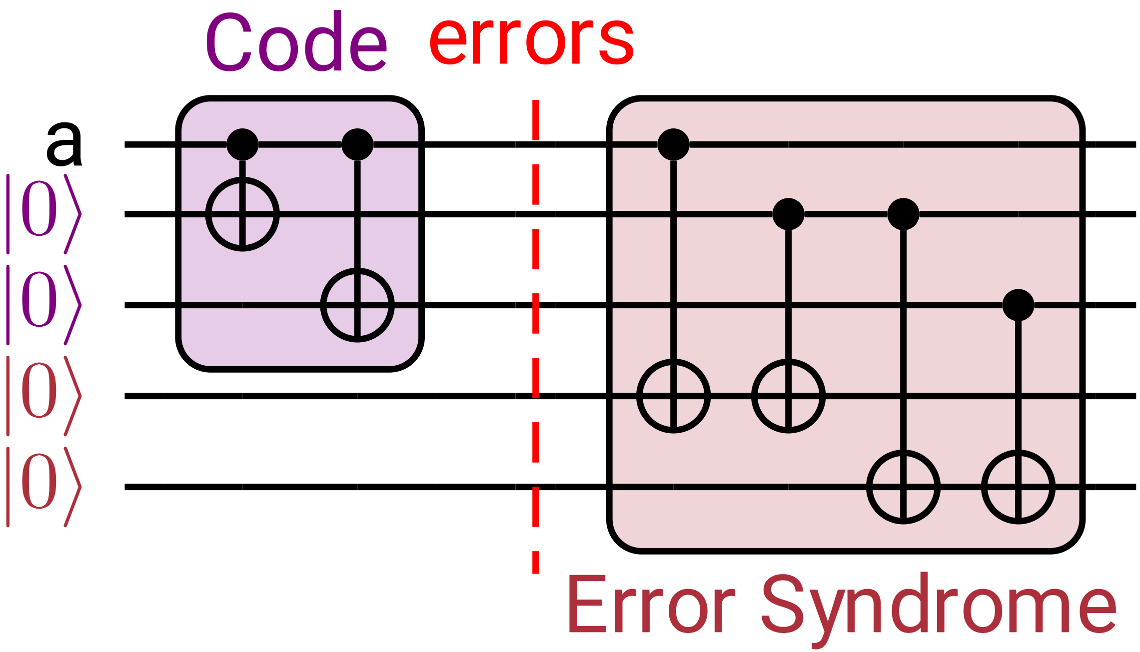
Input: $A = |0\rangle \otimes (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) \otimes |0\rangle = \alpha \cdot |000\rangle + \beta \cdot |010\rangle$
Error Syndrome:
$E = (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) \otimes (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) = \alpha^2 \cdot |00\rangle + \alpha \cdot \beta \cdot (|01\rangle + |10\rangle) + \beta^2 \cdot |11\rangle$

Quantum Repetition Code - Error Syndrome



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Incomplete bit flip:

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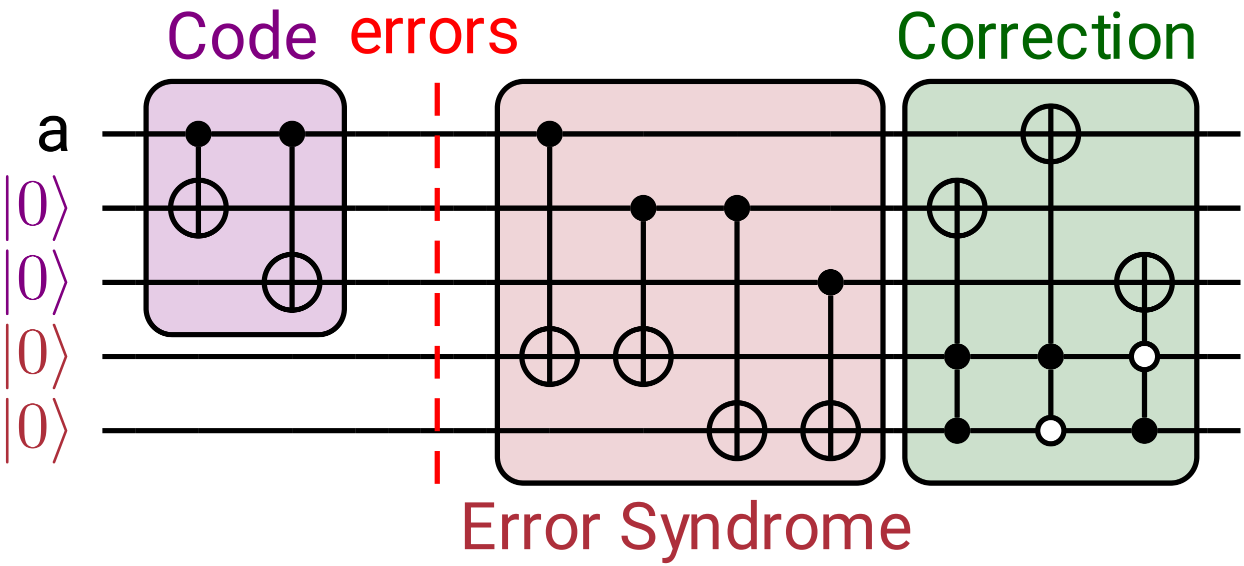
Error Syndrome:

$$E = (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) \otimes (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) = \alpha^2 \cdot |00\rangle + \alpha \cdot \beta \cdot (|01\rangle + |10\rangle) + \beta^2 \cdot |11\rangle$$

Action: Partially flip second qubit???

Quantum Repetition Code - Correcting Error



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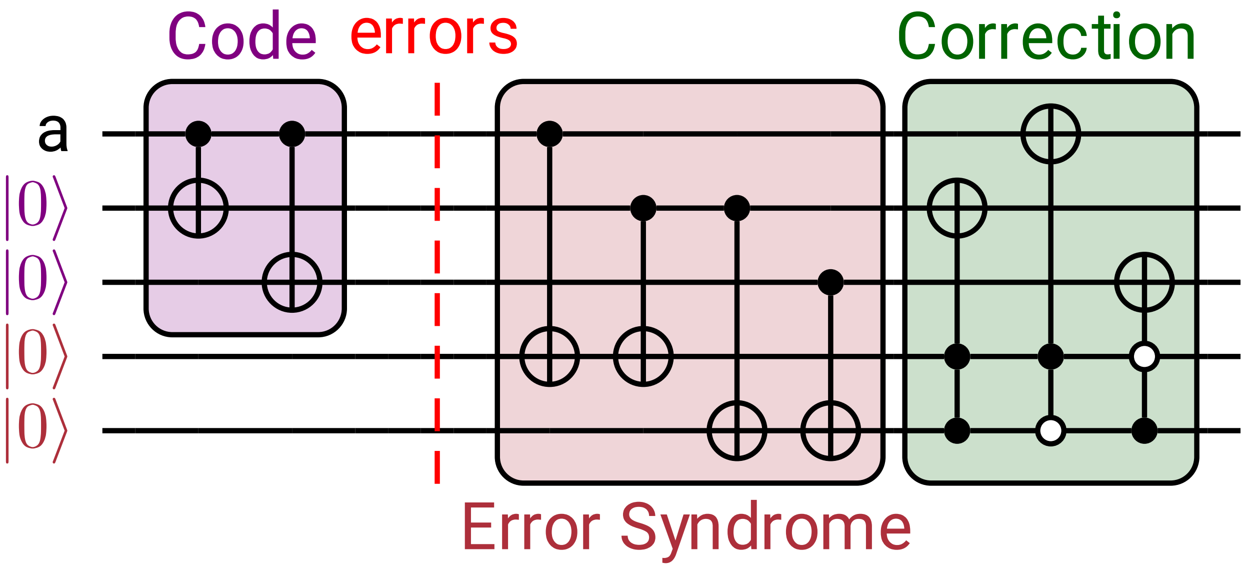


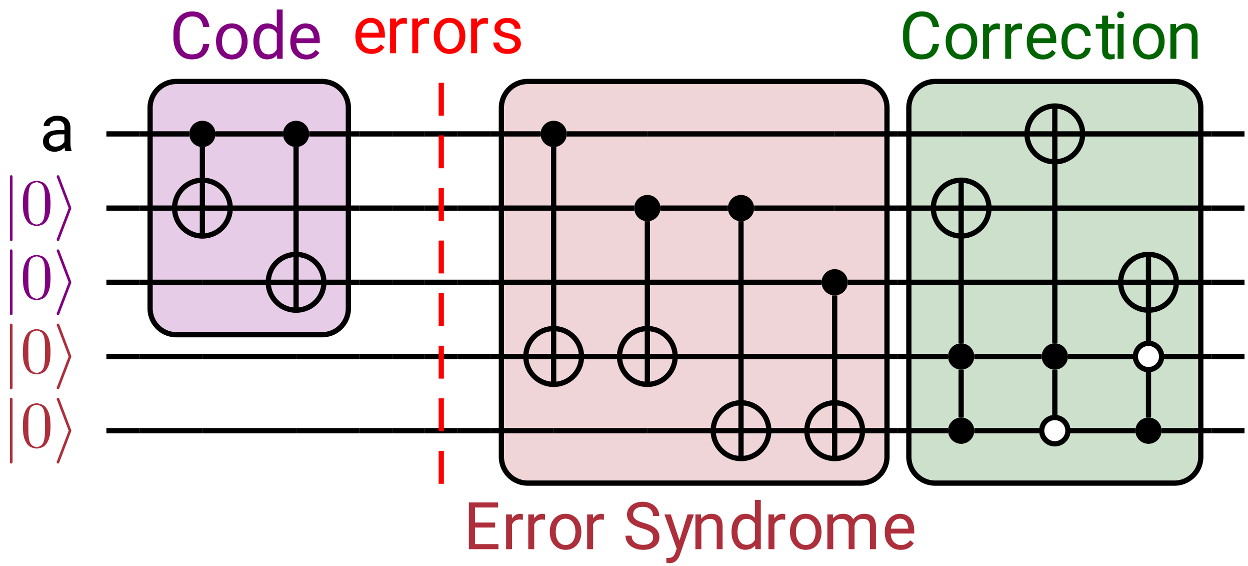
Input: $A \otimes E$

After first CCNOT:

$$CCNOT \cdot (A \otimes E) = A \otimes E - (\alpha \cdot \beta^2 \cdot |00011\rangle + \beta^3 \cdot |01011\rangle) + (\beta^3 \cdot |00011\rangle + \alpha \cdot \beta^2 \cdot |01011\rangle)$$

e.g., $\beta = 0.9, \alpha \approx 0.43589$, i.e. $\beta^3 = 0.729, \alpha \cdot \beta^2 \approx 0.353, \beta^3 - \alpha \cdot \beta^2 \approx 0.376$

Quantum Repetition Code - Correcting Error



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Second and third CCNOT also (slightly) change first and third qubit...

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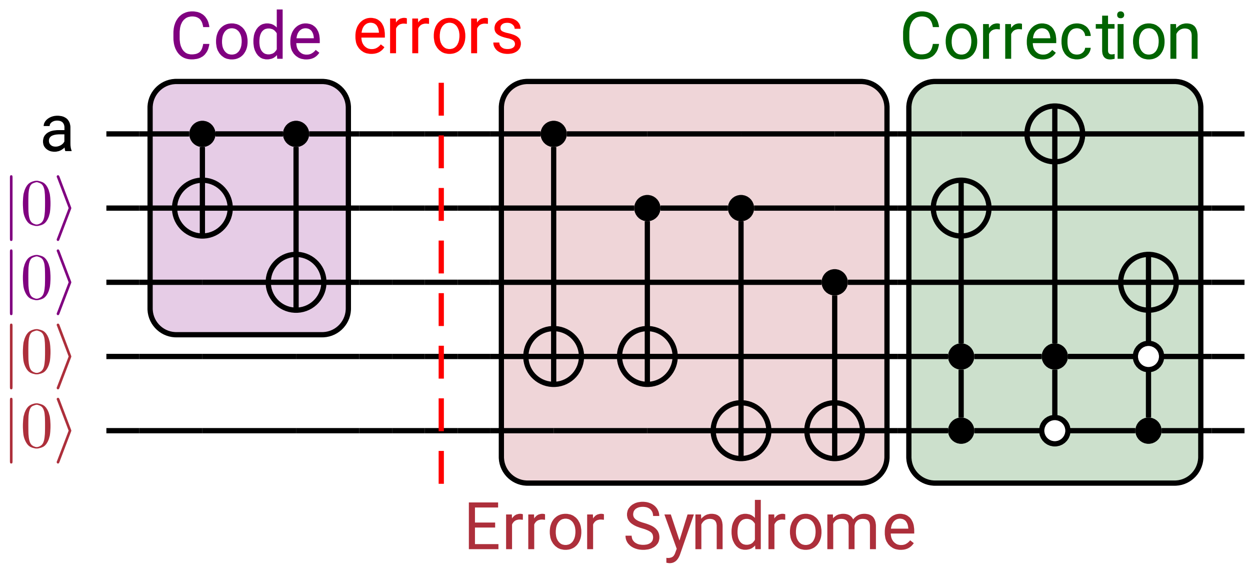

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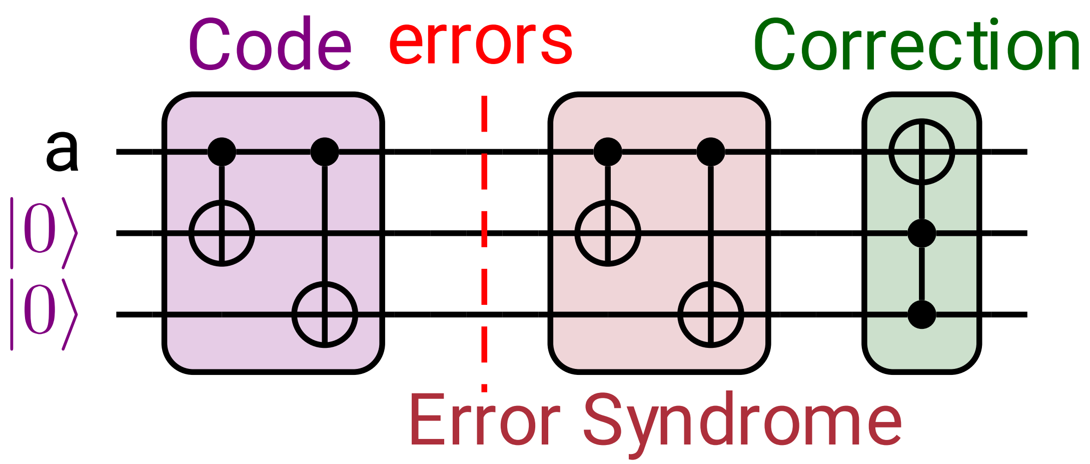
Second and third CCNOT also (slightly) change first and third qubit...

Here: correction of 3 repeated qubits.
How to correct only 1 qubit?

Quantum Repetition Code - Correcting Error



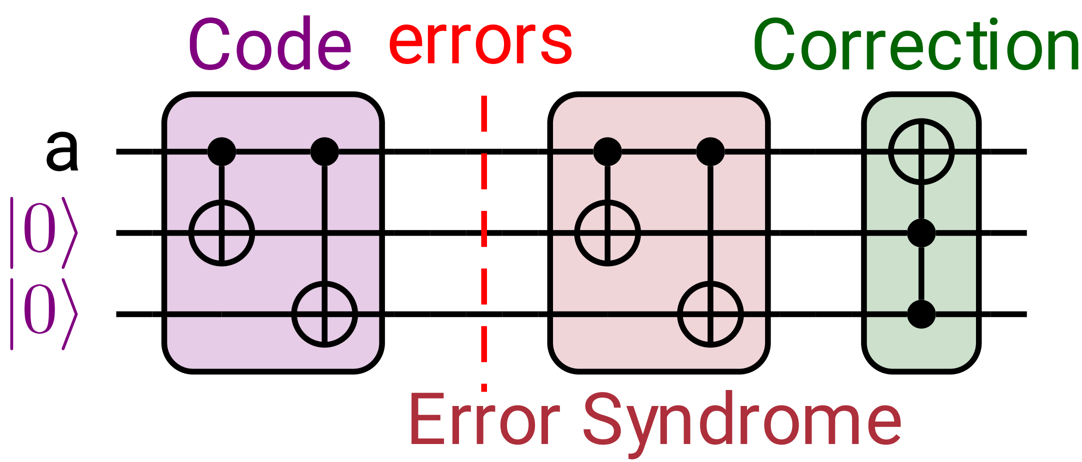
Input (with errors)	Error Syndrome	After Correction
$\|000\rangle$		
$\|001\rangle$		
$\|010\rangle$		
$\|011\rangle$		
$\|100\rangle$		
$\|101\rangle$		
$\|110\rangle$		
$\|111\rangle$		

Input (with errors): $\alpha|010\rangle + \beta|011\rangle$

Error Syndrome:

After Correction:

Quantum Repetition Code - Correcting Error



Input (with errors)	Error Syndrome	After Correction
$\|000\rangle$	$\|000\rangle$	$\|000\rangle$
$\|001\rangle$	$\|001\rangle$	$\|001\rangle$
$\|010\rangle$	$\|010\rangle$	$\|010\rangle$
$\|011\rangle$	$\|011\rangle$	$\|111\rangle$
$\|100\rangle$	$\|111\rangle$	$\|011\rangle$
$\|101\rangle$	$\|110\rangle$	$\|110\rangle$
$\|110\rangle$	$\|101\rangle$	$\|101\rangle$
$\|111\rangle$	$\|100\rangle$	$\|100\rangle$

Input (with errors): $\alpha|010\rangle + \beta|011\rangle$
Error Syndrome: $\alpha|010\rangle + \beta|011\rangle$
After Correction: $\alpha|010\rangle + \beta|111\rangle$

Michael A. Nielsen,Isaac L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2000

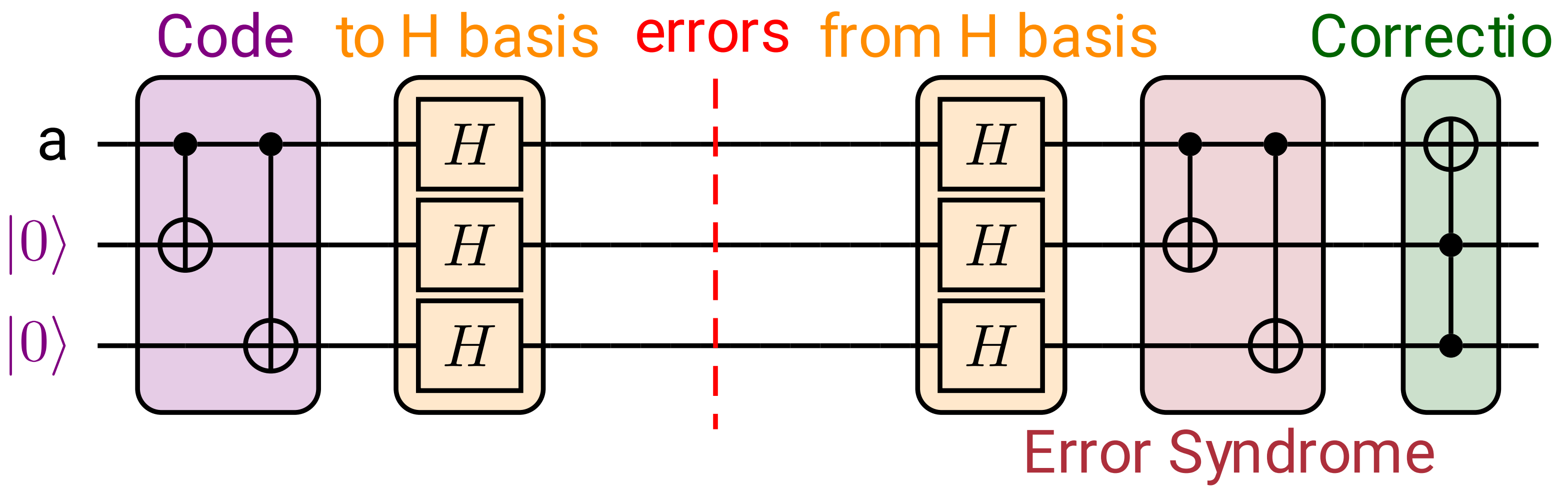
Correcting Phase Errors

- Hadamard transform H exchanges bit flip and phase errors
  - $H(\alpha \cdot |0\rangle + \beta \cdot |1\rangle) = \alpha \cdot |+\rangle + \beta \cdot |-\rangle$
- NOT X operator in Hadamard basis acts like a phase flip
  - $X \cdot |+\rangle = X \cdot \frac{|0\rangle+|1\rangle}{\sqrt{2}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$
  - $X \cdot |-\rangle = X \cdot \frac{|0\rangle-|1\rangle}{\sqrt{2}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = -|-\rangle$
- Pauli Z operator in Hadamard basis acts like a bit flip
  - $Z \cdot |+\rangle = Z \cdot \frac{|0\rangle+|1\rangle}{\sqrt{2}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = |-\rangle$
  - $Z \cdot |-\rangle = Z \cdot \frac{|0\rangle-|1\rangle}{\sqrt{2}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$

$\Rightarrow$ Repetition code in Hadamard basis

$\alpha \cdot |+\rangle + \beta \cdot |-\rangle \rightarrow \alpha \cdot |+++\rangle + \beta \cdot |---\rangle$ corrects a phase error!

Correcting Phase Errors



- also called Sign Flip Code

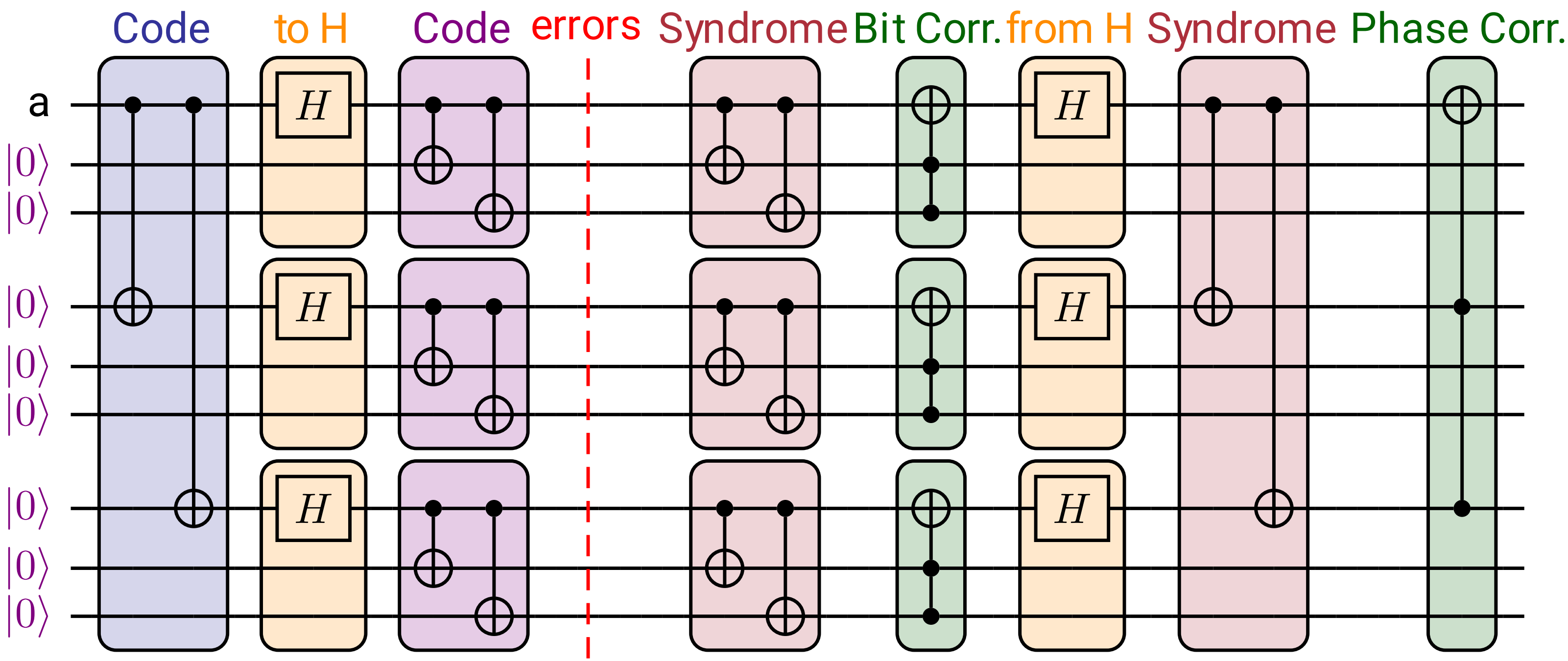
Correcting both Bit and Phase Flips: Shor Code

- Use bit flip code and sign flip code at once (using 9 qubits):
  $\alpha \cdot |0\rangle + \beta \cdot |1\rangle \rightarrow \alpha \cdot (|000\rangle + |111\rangle)^{\otimes 3} + \beta \cdot (|000\rangle - |111\rangle)^{\otimes 3}$

Correction of	by
Bit Flip Error	Repetition of Bit: 000, 111
Phase/Sign Flip Error	Repetition of Phase: $+++$, $---$

- Arbitrary error can be modeled by a unitary transform $U = c_0 \cdot I + c_1 \cdot X + c_2 \cdot Y + c_3 \cdot Z$, where $c_0, \cdots, c_3$ are complex constants
  - Bit Flip Error: $U = X$
  - Phase/Sign Flip Error: $U = Z$
  - Both Bit and Phase/Sign Flip Errors: $U = i \cdot Y$, where $Y = i \cdot X \cdot Z$
- Shor code can correct arbitrary 1-qubit errors due to linearity [G'09]
  - For small errors $\epsilon \cdot (a_X \cdot X + a_Y \cdot Y + a_Z \cdot Z)$, where $a_i \in \{0, 1\}$:
    After error correction the quantum state will be correct within $O(\epsilon^2)$ [G'09]

Quantum circuit of the Shor code



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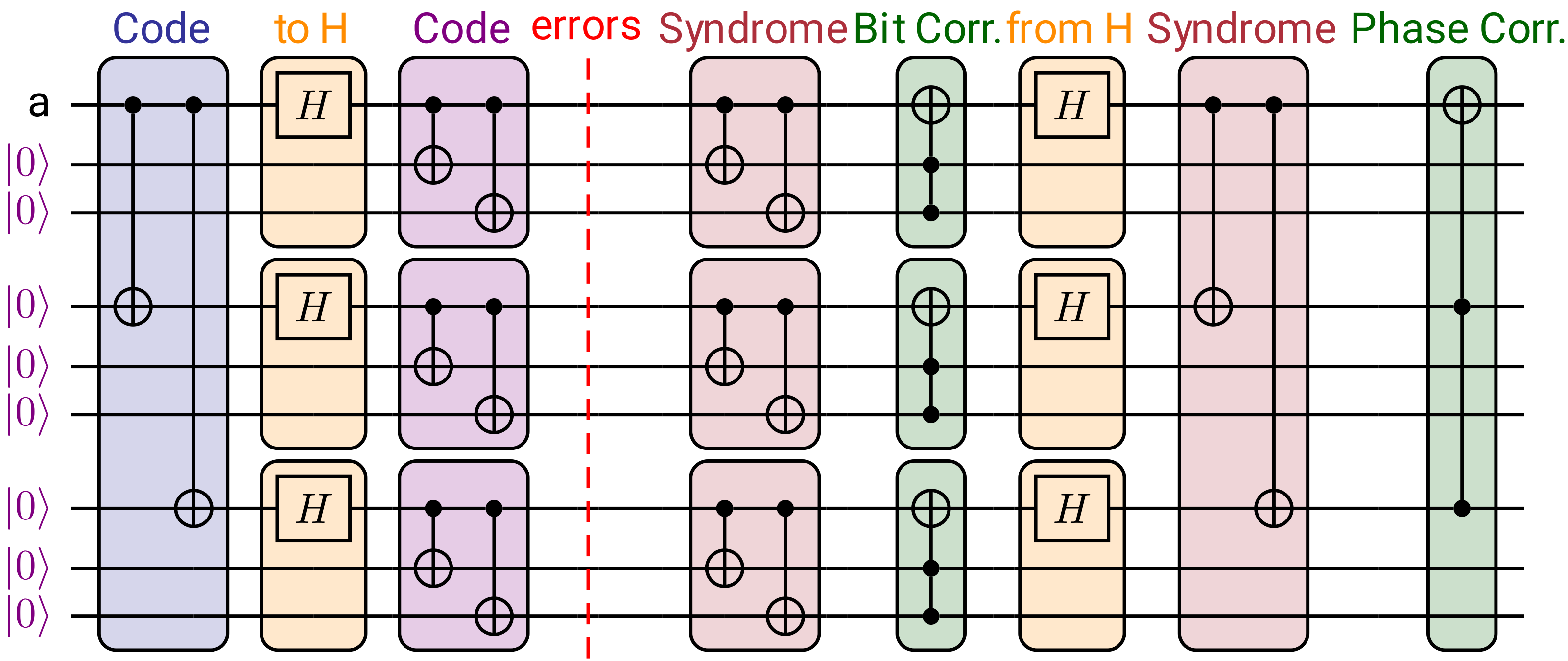

- There are good codes with less qubits
  - e.g., 5-Qubit Code
    - smallest quantum error correcting code protecting a logical qubit from any arbitrary single qubit error

Advanced codes

- rely on assemblies of physical qubits
  - that are called logical qubits, and
  - are extraordinarily tolerant to local physical qubit errors
- Examples
  - Surface code [BK98] [FSG09] $\rightsquigarrow$ Google
    - a number of variants that are typically placed in a two-dimensional topology
  - Low-Density Parity-Check (LDPC) code $\rightsquigarrow$ IBM
    - designed for classical computing [G62]
      - capacity-approaching code, i.e., practical constructions exist that allow the noise threshold to be set very close to the theoretical maximum (called Shannon limit)
      - can be decoded in time linear to their block length
    - several quantum variants including quantum Tanner codes [PK22][LZ22][B+23]
      - with $15\times$ less encoding overhead compared with surface codes,
      - not locally embeddable into a 2D grid, but can be decomposed into two planar degree-3 subgraphs well-suited for architectures based on superconducting qubits

Quantum Threshold Theorem

(also called Quantum Fault-Tolerance Theorem)

A quantum circuit on n qubits and containing $p(n)$ gates may be simulated with probability of error at most ε using $O\left(\log^c\left(\frac{p(n)}{\varepsilon}\right)\cdot p(n)\right)$ gates (for some constant c) on hardware whose components fail with probability at most p, provided p is below some constant threshold, $p < p_{th}$, and given reasonable assumptions about the noise in the underlying hardware.

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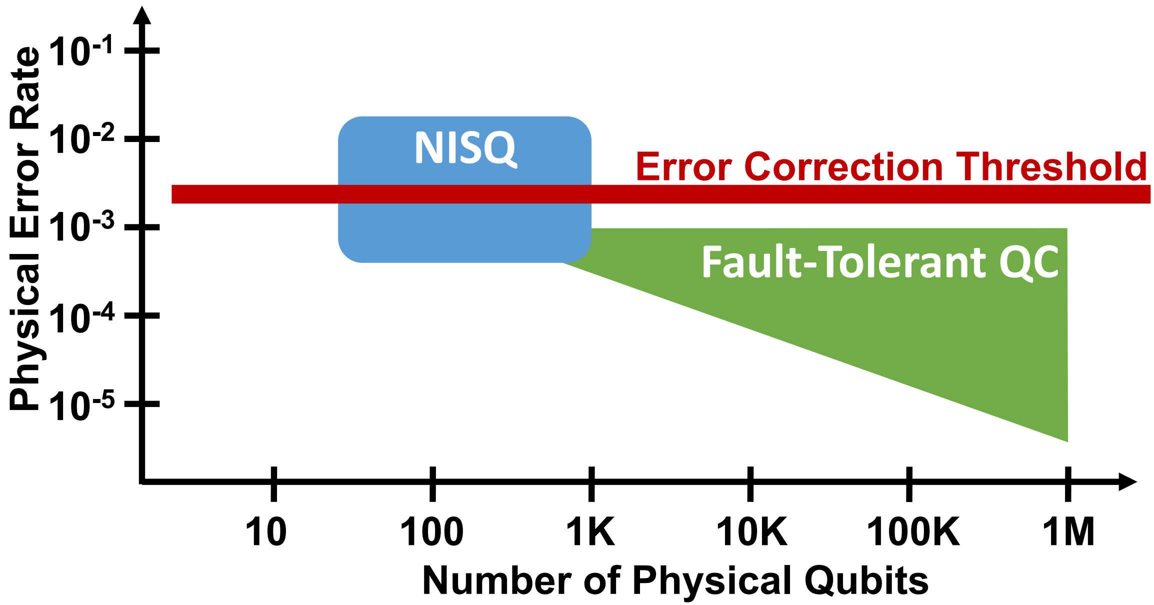
- $\Rightarrow$ quantum computers can be made fault-tolerant
- Result was proven (for various error models) [AB'08] [KLZ'98] [K'03] [S'96], in experiments suppressing logical errors by scaling a quantum error-correcting code [A+'23]
- *"The entire content of the Threshold Theorem is that you're correcting errors faster than they're created. That's the whole point, and the whole non-trivial thing that the theorem shows. That's the problem it solves."* [AG'06]

Noisy Intermediate-Scale Quantum (NISQ)



- quantum computers with 50-100 qubits: noise in quantum gates limits the size of quantum circuits that can be executed reliably
- Such NISQ devices may be able to perform tasks which surpass the capabilities of today’s classical digital computers with application areas like quantum chemistry, optimization & machine learning
- Today's 100(0)-qubit QPUs are (only) intermediate technologies

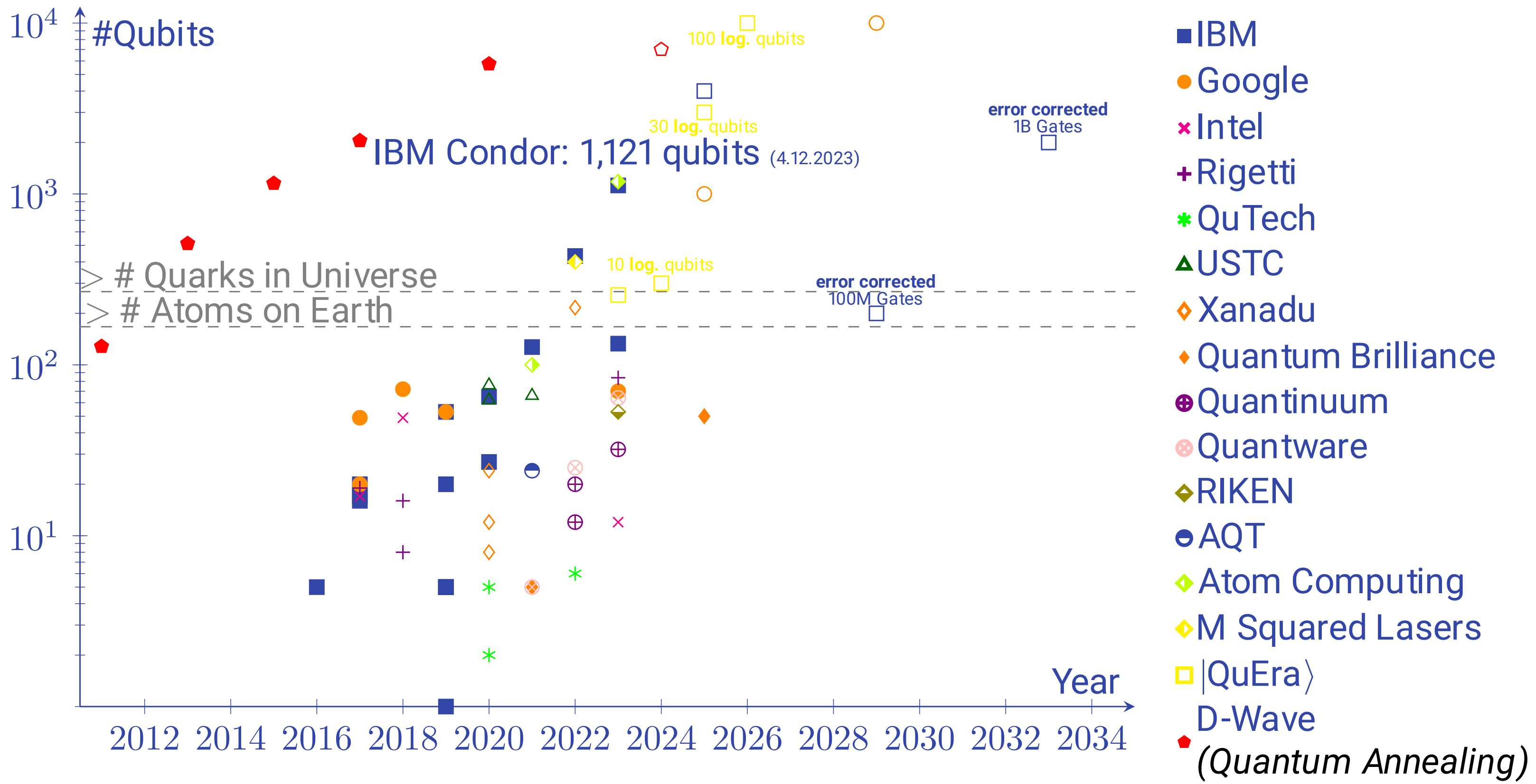
Threshold for Surface and LDPC code

- Surface Code $\rightsquigarrow$ Google:
  - Threshold: $\approx$ 1% [FSG'13]
    $\Rightarrow\approx$ **1,000-10,000 physical qubits per logical data qubit** for the surface code at a 0.1% probability of a depolarizing error [FMC'12][CTV'17]
  - $\varepsilon_X :=$ logical error rate of distance X surface code, today $\frac{\epsilon_3}{\epsilon_5} = 1.04$.
    $\frac{\varepsilon_3}{\varepsilon_{17}} = 4$ using 577 physical qubits $\Rightarrow$ logical error rate $<$ 1 in 10^6 [NG'23] [A+'23] (i.e., experimental demonstration in which quantum error correction begins to improve performance with increasing qubit number)
- LDPC Code [B+23] $\rightsquigarrow$ IBM:
  - requires a better short range CNOT connectivity between qubits (in IBM's roadmap thanks to a multilayer connectivity chipset)
  - For a logical qubit error rate of 10^{-12}:
    12 logical qubits would require 288 physical qubits (instead of 4,000 physical qubits with a surface code), assuming a physical qubit error rate of 0.1% (which remains to be seen at such scales)
  - 2029 first quantum computer with quantum error correction [QS'23]

Further exciting news

- Logical quantum processor based on reconfigurable atom arrays
  - December 2023: Using three-dimensional [[8,3,2]] code blocks, researchers from Harvard, MIT and QuEra Computing realized computationally complex sampling circuits with up to 48 logical qubits entangled with hypercube connectivity with 228 logical two-qubit gates and 48 logical CCZ gates [B+23]
- Quantinuum
  - April 3rd, 2024: 4 logical qubits encoded into just 30 physical qubits on H2 quantum computer [S+24]
  - April 16th, 2024: first demonstration of 99.914(3)% 2-qubit gate fidelity without error correction in H1-1 quantum computer
    - ~1000 entangling operations can be done before an error occurs

Timeline of #Qubits



Summary & Conclusions

- Error Correction
  - Classical Repetition Code
  - Barriers to Quantum Error Correction
  - Quantum Repetition Code
    - Correcting Bit Flips
    - Correcting Phase/Sign Flips
  - Shor Code (using 9 qubits)
    - correcting arbitrary 1-qubit errors
  - still codes with less qubits (minimum 5 qubits)
- Quantum Threshold (/Fault-Tolerance) Theorem
  - fault-tolerant quantum computers are possible
    - IBM: 2029
  - surface code: 1,000-10,000 physical qubits per logical data qubit
  - LDPC code: 288 physical qubits for 12 logical qubits