Lecture

Quantum Computing
(CS5070)
Quantum Error Correction

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Error Correction

- **Physical devices are imperfect**
  - for quantum devices even more true than for classical devices
- **Interactions with the environment**
  - makes errors (like decoherence for quantum devices) more likely
- **Error must be controlled or compensated**
  - Probability of one step to succeed: \( p \)
  - Probability of \( t \) steps to succeed: \( p^t \)
Quantum Error

- A quantum error can occur at any time
  - changing the superposition $\alpha |0\rangle + \beta |1\rangle$ of a qubit to $\alpha' |0\rangle + \beta' |1\rangle$
  - quantum errors can occur in several qubits at the same time
- **Examples of single-qubit errors:**

| Name               | Operator | Effect on $|0\rangle$ | Effect on $|1\rangle$ |
|--------------------|----------|-----------------------|------------------------|
| Bit Flip           | $X$      | $X |0\rangle = |1\rangle$ | $X |1\rangle = |0\rangle$ |
| Phase Flip         | $Z$      | $Z |0\rangle = |0\rangle$ | $Z |1\rangle = -|1\rangle$ |
| Rotation           | $R_\theta$ | $R_\theta |0\rangle = |0\rangle$ | $R_\theta |1\rangle = e^{i\theta} |1\rangle$ |
| (Full) Decoherence |          | $\alpha |0\rangle + \beta |1\rangle \rightarrow \{ |0\rangle, |1\rangle \}$ |
Decoherence

- [SAG’19] runs amplitude and phase damping simulations
  - Decoherence (of amplitude and phase) depends on
    - type of gate
    - input
    - quantum circuit depth
  - simple (logical) optimizations resulting in lower-depth circuits can improve the decoherence by 20%
    - comparing full adders with depths 6 and 9

<table>
<thead>
<tr>
<th>Input</th>
<th>Gates/Circuits from lowest to highest (amplitude) decoherence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>000\rangle +</td>
</tr>
<tr>
<td>$\sqrt{2^3}$</td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>110\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>101\rangle$</td>
</tr>
<tr>
<td>Any input</td>
<td>Full adders (3 qubits $\rightarrow$ sum+carry):</td>
</tr>
<tr>
<td></td>
<td>QCKT-1 (depth 6), QCKT-2 (depth 9)</td>
</tr>
</tbody>
</table>
(Simple) **Classical Repetition Code**

- **Classical data:**
  - **Repetition code** for correction of a single bit-flip:
    - $0 \rightarrow 000$
    - $1 \rightarrow 111$
  - **A single bit flip error**
    - Choosing the majority of the three bits, e.g. $010 \rightarrow 0$
  - In case of rare errors: 1 error is more likely than 2
(Simple) **Classical Repetition Code**

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- **How to adapt** this approach for quantum error correction?
Barriers to Quantum Error Correction

- Measurement destroys superpositions
- No-cloning theorem prevents repetition
- Correction of multiple types of errors
  - e.g., bit flip and phase errors
- How to correct continuous errors and decoherence?
Barriers to Quantum Error Correction

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- Higher circuit depths due to quantum error correction approaches
  - Can state-of-the-art quantum computers process these circuit depths?
Quantum Repetition Code correcting bit flip errors

- **How to construct quantum circuit** for quantum repetition code?
  - $|0\rangle \rightarrow |000\rangle$
  - $|1\rangle \rightarrow |111\rangle$
  - $\alpha |0\rangle + \beta |1\rangle \rightarrow \alpha |000\rangle + \beta |111\rangle$
Quantum Repetition Code - Generate Code

Next step: How to detect qubit flips without measurement and cloning?

\[ \alpha |00\rangle + \beta |11\rangle \rightarrow \alpha |000\rangle + \beta |111\rangle \]
Quantum Repetition Code - Error Syndrome
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<table>
<thead>
<tr>
<th>Input</th>
<th>Error Syndrome</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>00\rangle</td>
<td>None</td>
</tr>
<tr>
<td>111\rangle</td>
<td>00\rangle</td>
<td>Flip 2nd qubit</td>
</tr>
<tr>
<td>010\rangle</td>
<td>11\rangle</td>
<td>Flip 1st qubit</td>
</tr>
<tr>
<td>101\rangle</td>
<td>10\rangle</td>
<td>Flip 3rd qubit</td>
</tr>
<tr>
<td>100\rangle</td>
<td>01\rangle</td>
<td>Flip 3rd qubit</td>
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</tbody>
</table>
Quantum Repetition Code - Error Syndrome

Code errors

Error Syndrome

Incomplete bit flip:
Input: \( A = |0\rangle \otimes (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) \otimes |0\rangle = \alpha \cdot |000\rangle + \beta \cdot |010\rangle \)
Quantum Repetition Code - Error Syndrome

Incomplete bit flip:
Input: \( A = |0\rangle \otimes (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) \otimes |0\rangle = \alpha \cdot |000\rangle + \beta \cdot |010\rangle \)
Error Syndrome: \( E = (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) \otimes (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) = \alpha^2 \cdot |00\rangle + \alpha \cdot \beta \cdot (|01\rangle + |10\rangle) + \beta^2 \cdot |11\rangle \)
Quantum Repetition Code - Error Syndrome

Code errors

| 0 ⟩
| 0 ⟩
| 0 ⟩
| 0 ⟩

Error Syndrome

\[
\text{Incomplete bit flip:} \\
\text{Input: } A = |0\rangle \otimes (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) \otimes |0\rangle = \alpha \cdot |000\rangle + \beta \cdot |010\rangle \\
\text{Error Syndrome: } E = (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) \otimes (\alpha \cdot |0\rangle + \beta \cdot |1\rangle) = \alpha^2 \cdot |00\rangle + \alpha \cdot \beta \cdot (|01\rangle + |10\rangle) + \beta^2 \cdot |11\rangle \\
\text{Action: Partially flip second qubit??}
\]
Quantum Repetition Code - Correcting Error

Code errors

Correction

Error Syndrome
Quantum Repetition Code - Correcting Error

Input: $A \otimes E$

After first CCNOT: $CCNOT \cdot (A \otimes E) = A \otimes E - (\alpha \cdot \beta^2 \cdot |00011\rangle + \beta^3 \cdot |01011\rangle) + (\beta^3 \cdot |00011\rangle + \alpha \cdot \beta^2 \cdot |01011\rangle)$

e.g., $\beta = 0.9, \alpha \approx 0.43589$, i.e. $\beta^3 = 0.729, \alpha \cdot \beta^2 \approx 0.353, \beta^3 - \alpha \cdot \beta^2 \approx 0.376$
Quantum Repetition Code - Correcting Error

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Second and third CCNOT also (slightly) change first and third qubit...
Quantum Repetition Code - Correcting Error

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Second and third CCNOT also (slightly) change first and third qubit...

Here: correction of 3 repeated qubits.

How to correct only 1 qubit?
Quantum Repetition Code - Correcting Error

Input (with errors): $\alpha|010\rangle + \beta|011\rangle$

Error Syndrome: After Correction:
Quantum Repetition Code - Correcting Error

Input (with errors) | Error Syndrome | After Correction
--- | --- | ---
000 | 000 | 000
001 | 001 | 001
010 | 010 | 010
011 | 011 | 111
100 | 111 | 011
101 | 110 | 110
110 | 101 | 101
111 | 100 | 100

Input (with errors): $\alpha|010\rangle + \beta|011\rangle$
Error Syndrome: $\alpha|010\rangle + \beta|011\rangle$
After Correction: $\alpha|010\rangle + \beta|111\rangle$
Correcting Phase Errors

- Hadamard transform $H$ exchanges bit flip and phase errors
  $$H(\alpha \cdot |0\rangle + \beta \cdot |1\rangle) = \alpha \cdot |+\rangle + \beta \cdot |--\rangle$$

- NOT $X$ operator in Hadamard basis acts like a phase flip
  $$X \cdot |+\rangle = X \cdot |0\rangle \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |+\rangle$$
  $$X \cdot |--\rangle = X \cdot |1\rangle \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |--\rangle$$

- Pauli $Z$ operator in Hadamard basis acts like a bit flip
  $$Z \cdot |+\rangle = Z \cdot |0\rangle \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = |--\rangle$$
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$\Rightarrow$ Repetition code in Hadamard basis $\alpha \cdot |+\rangle + \beta \cdot |--\rangle \rightarrow \alpha \cdot |++\rangle + \beta \cdot |--\rangle$ corrects a phase error!
Correcting Phase Errors

- also called Sign Flip Code
Correcting both **Bit and Phase Flips:**

**Shor Code**

- Use bit flip code and sign flip code at once (using 9 qubits):
  \[\alpha \cdot |0\rangle + \beta \cdot |1\rangle \rightarrow \alpha \cdot (|000\rangle + |111\rangle) \otimes 3 + \beta \cdot (|000\rangle - |111\rangle) \otimes 3\]
  
  **Correction of Bit Flip Error**
  - by Repetition of Bit: 000, 111
  
  **Correction of Phase/Sign Flip Error**
  - by Repetition of Phase: ++ +, − − −

- Arbitrary error can be modeled by a unitary transform \[U = c_0 \cdot I + c_1 \cdot X + c_2 \cdot Y + c_3 \cdot Z,\]
  where \(c_0, \cdots, c_3\) are complex constants
  - Bit Flip Error: \(U = X\)
  - Phase/Sign Flip Error: \(U = Z\)
  - Both Bit and Phase/Sign Flip Errors: \(U = i \cdot Y, \text{where} Y = i \cdot X \cdot Z\)

- Shor code can correct arbitrary 1-qubit errors due to linearity [G'09]
  - For small errors \(\epsilon \cdot (a_X \cdot X + a_Y \cdot Y + a_Z \cdot Z), \text{where} a_i \in \{0, 1\}\):
    After error correction the quantum state will be correct within \(O(\epsilon^2)\) [G'09]
Quantum circuit of the Shor code
Quantum circuit of the Shor code

- There are good codes with less qubits
  - e.g., 5-Qubit Code
    - smallest quantum error correcting code protecting a logical qubit from any arbitrary single qubit error
Advanced codes

- **rely on assemblies of physical qubits**
  - that are called logical qubits, and
  - are extraordinarily tolerant to local physical qubit errors

- **Examples**
  - **Surface code** [BK98] [FSG09] \(\rightsquigarrow\) Google
    - a number of variants that are typically placed in a two-dimensional topology
  - **Low-Density Parity-Check (LDPC) code** \(\rightsquigarrow\) IBM
    - designed for classical computing [G62]
      - capacity-approaching code, i.e., practical constructions exist that allow the noise threshold to be set very close to the theoretical maximum (called Shannon limit)
      - can be decoded in time linear to their block length
    - several quantum variants including quantum Tanner codes [PK22][LZ22][B+23]
      - with \(15\times\) less encoding overhead compared with surface codes,
      - not locally embeddable into a 2D grid, but can be decomposed into two planar degree-3 subgraphs well-suited for architectures based on superconducting qubits
Quantum Threshold Theorem
(also called Quantum Fault-Tolerance Theorem)

A quantum circuit on $n$ qubits and containing $p(n)$ gates may be simulated with probability of error at most $\varepsilon$ using $O\left(\log^{c} \left(\frac{p(n)}{\varepsilon}\right) \cdot p(n)\right)$ gates (for some constant $c$) on hardware whose components fail with probability at most $p$, provided $p$ is below some constant threshold, $p < p_{th}$, and given reasonable assumptions about the noise in the underlying hardware.
Quantum Threshold Theorem
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A quantum circuit on \( n \) qubits and containing \( p(n) \) gates may be simulated with probability of error at most \( \varepsilon \) using \( O \left( \log^c \left( \frac{p(n)}{\varepsilon} \right) \cdot p(n) \right) \) gates (for some constant \( c \)) on hardware whose components fail with probability at most \( p \), provided \( p \) is below some constant threshold, \( p < p_{th} \), and given reasonable assumptions about the noise in the underlying hardware.

- quantum computers can be made fault-tolerant
- Result was proven (for various error models) [AB'08] [KLZ'98] [K'03] [S'96], in experiments suppressing logical errors by scaling a quantum error-correcting code [A+'23]
- "The entire content of the Threshold Theorem is that you're correcting errors faster than they're created. That's the whole point, and the whole non-trivial thing that the theorem shows. That's the problem it solves." [AG'06]
Noisy Intermediate-Scale Quantum (NISQ)

- quantum computers with 50-100 qubits: noise in quantum gates limits the size of quantum circuits that can be executed reliably
- Such NISQ devices may be able to perform tasks which surpass the capabilities of today’s classical digital computers with application areas like quantum chemistry, optimization & machine learning
- Today's 100(0)-qubit QPUs are (only) intermediate technologies
Threshold for Surface and LDPC code

- **Surface Code** $\rightsquigarrow$ Google:
  
  - Threshold: $\approx 1\%$ \cite{FSG13}

    \[ \Rightarrow 1,000-10,000 \text{ physical qubits per logical data qubit} \] for the surface code at a 0.1% probability of a depolarizing error \cite{FMC12,CTV17}

    - $\varepsilon_X$ := logical error rate of distance $X$ surface code, today $\frac{\varepsilon_3}{\varepsilon_5} = 1.04$

      \[ \frac{\varepsilon_3}{\varepsilon_{17}} = 4 \text{ using 577 physical qubits} \Rightarrow \text{logical error rate < 1 in 10}^{16} \text{ \cite{NG23, A+23} (i.e.,}

      experimental demonstration in which quantum error correction begins to improve performance with increasing qubit number)

  
  - **LDPC Code** \cite{B+23} $\rightsquigarrow$ IBM:

    - requires a better short range CNOT connectivity between qubits (in IBM’s roadmap thanks to a multilayer connectivity chipset)

    - For a logical qubit error rate of $10^{-12}$:

      **12 logical qubits would require 288 physical qubits** (instead of 4,000 physical qubits with a surface code), assuming a physical qubit error rate of 0.1% (which remains to be seen at such scales)

    - 2029 first quantum computer with quantum error correction \cite{QS23}
Timeline of #Qubits

IBM Condor: 1,121 qubits (4.12.2023)

- IBM
- Google
- Intel
- Rigetti
- QuTech
- USTC
- Xanadu
- Quantum Brilliance
- Quantinuum
- Quantware
- RIKEN
- AQT
- Atom Computing
- M Squared Lasers
- D-Wave

(Quantum Annealing)
Summary & Conclusions

- **Error Correction**
  - Classical Repetition Code
  - Barriers to Quantum Error Correction
  - Quantum Repetition Code
    - Correcting Bit Flips
    - Correcting Phase/Sign Flips
  - Shor Code (using 9 qubits)
    - correcting arbitrary 1-qubit errors
  - still codes with less qubits (minimum 5 qubits)

- **Quantum Threshold (Fault-Tolerance) Theorem**
  - fault-tolerant quantum computers are possible
    - IBM: 2029
  - surface code: 1,000-10,000 physical qubits per logical data qubit
  - LDPC code: 288 physical qubits for 12 logical qubits