Lecture

Quantum Computing

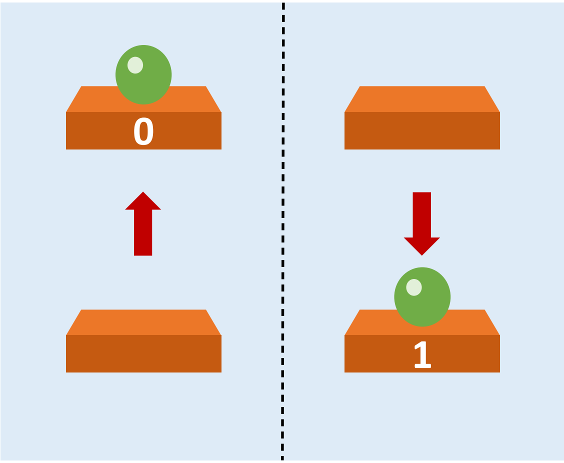
(CS5070)

Introduction to the Bloch Sphere

Professor Dr. rer. nat. habil. Sven Groppe
https://www.ifis.uni-luebeck.de/index.php?id=groppe

Classical System: 2 Level System with only two states with values 0 and 1

- Bit (short for **bi**nary digi**t**)
  - Smallest unit of information with values 0 and 1
  - Abstraction from physical realization, Bits can be realized in different ways (e.g. different levels of voltage, in main memory, on disk, SSD, different types of internet connection etc.)



- Illustration of a classic bit. The two possible states 0 and 1 of the bit are represented
  - by the position of a ball on the upper or lower shelf, or
  - by the orientation of a vector (upwards or downwards)
- Other characteristics like size and color of the ball are not important for the states

Quantum Mechanical Systems 1/2

- **Quantum Bit** (Qubit): Information unit with two states:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

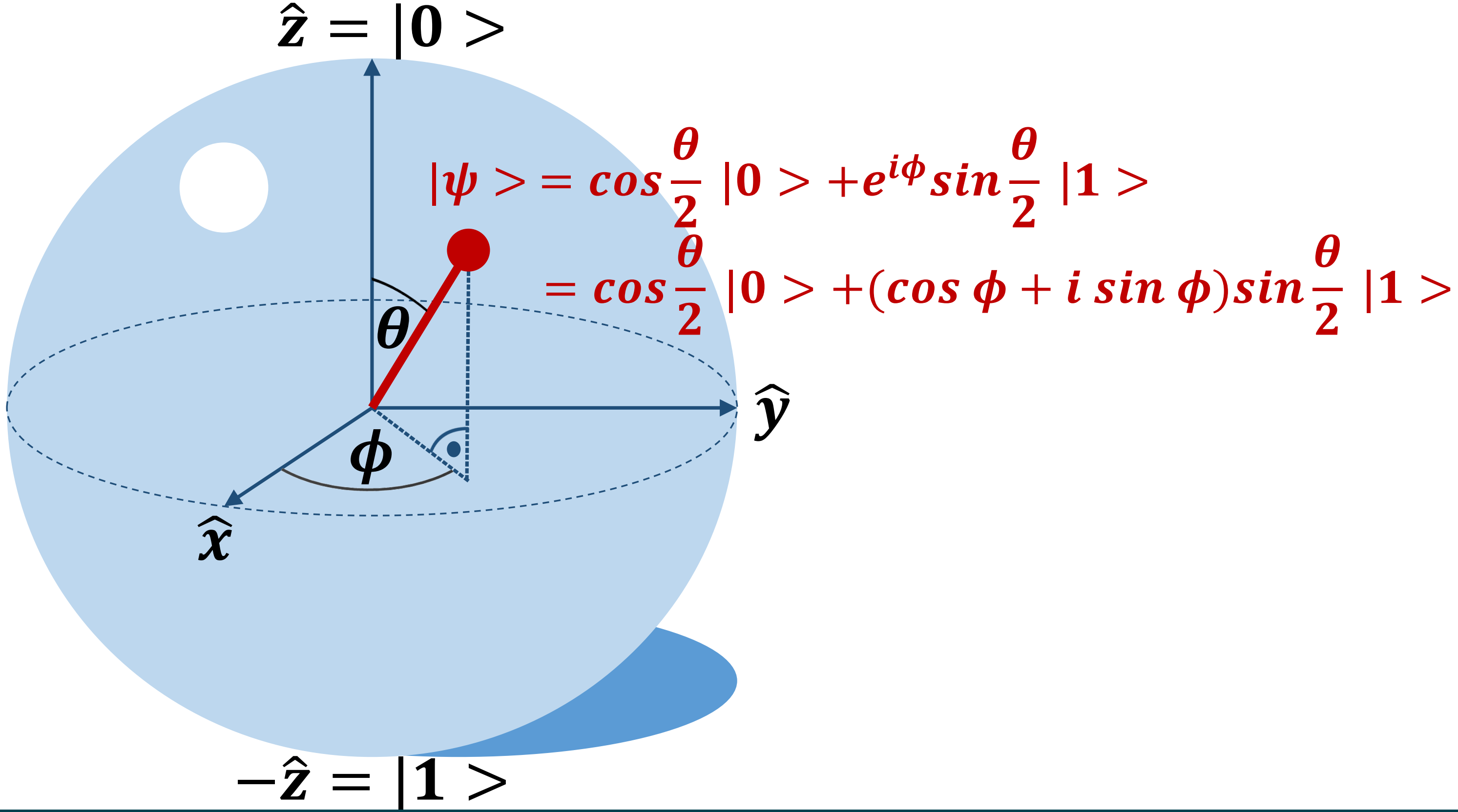
- Like for bits:
  - Abstraction from physical realization, Qubits can be realized in different ways (later in this lecture unit!)

Quantum Mechanical Systems 2/2

- Different from bits:
  - Superpositions of qubits are possible mathematical representations as sum of the two possible states with weighted complex amplitudes/complex coefficients α and β: $|\psi >= \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$
  - Restriction to normalized quantum states: $|\alpha|^2 + |\beta|^2 = 1$ in order to guarantee an interpretation of measurements with the meaning of probabilities: $P(|0\rangle) = |\alpha|^2$ and $P(|1\rangle) = |\beta|^2$
  - Any overall factor γ on a state for which $|\gamma| = 1$ is a 'global phase'. States that differ only by a global phase are physically indistinguishable.
  - Global phase is not relevant for all observable quantities of the quantum states $\Rightarrow$ α can be chosen as real number and not as a complex one
  - This superposition can be interpreted physically as an interference of the states

Representation of a Qubit in Bloch-Sphere

- Angels θ and ϕ can be associated with spherical coordinates on the so-called Bloch-sphere:



$$|\psi> = cos\frac{\theta}{2}|0> + e^{i\phi} sin\frac{\theta}{2}|1>$$

$$= cos\frac{\theta}{2}|0> + (cos\,\phi + i\,sin\,\phi) sin\frac{\theta}{2}|1>$$

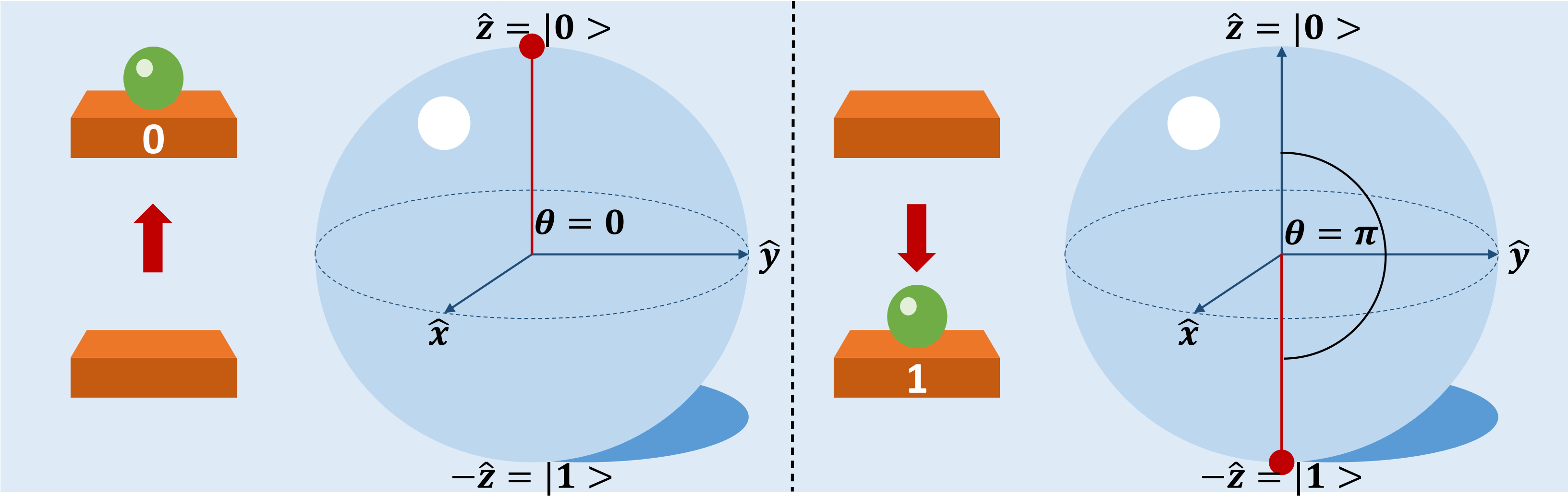
Representation of $|0\rangle$ and $|1\rangle$ in Bloch-Sphere

Classical System: Bit 0

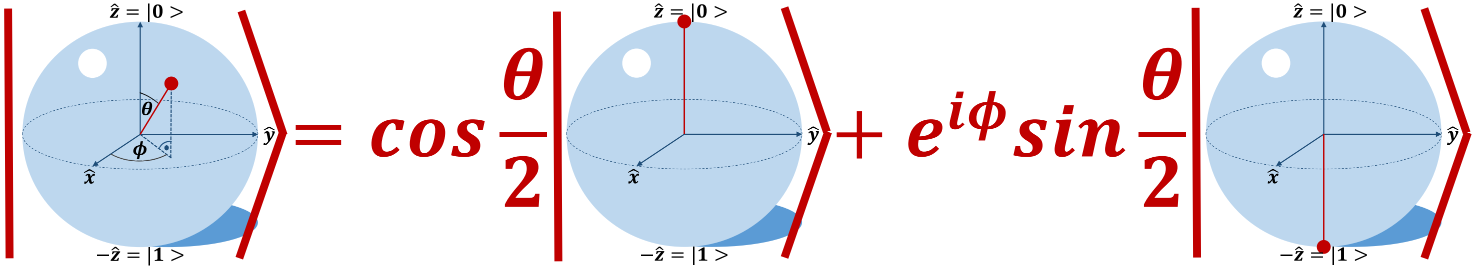
Quantum Mechanical System: Qubit 0

Classical System: Bit 1

Quantum Mechanical System: Qubit 1

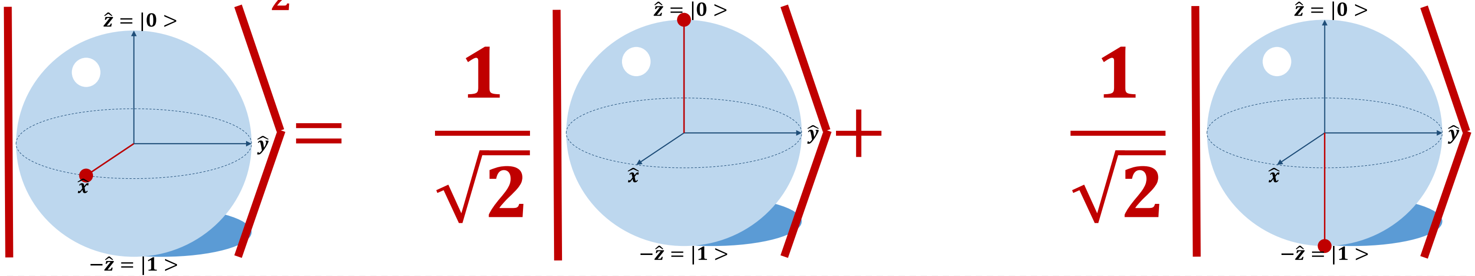


Superpositions in Bloch-Sphere



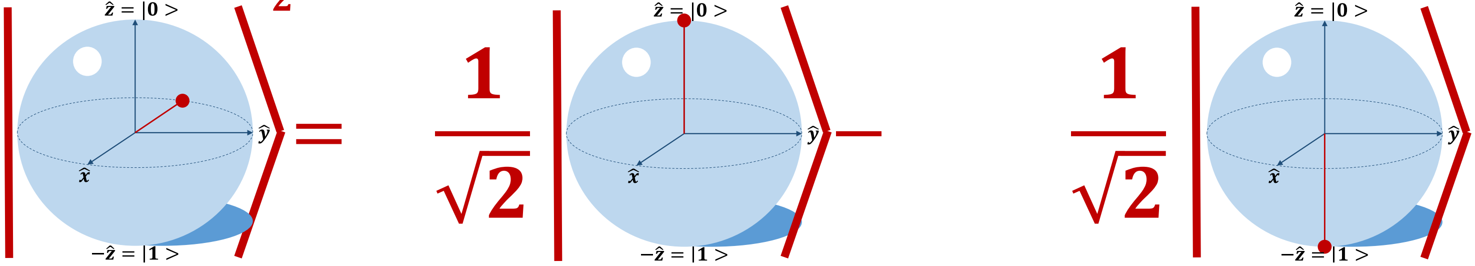
$$\left|\;\right\rangle = \cos\frac{\theta}{2}\left|\;\right\rangle + e^{i\phi}\sin\frac{\theta}{2}\left|\;\right\rangle$$

$|0_x > with\ \theta = \frac{\pi}{2}, \phi = 0:$



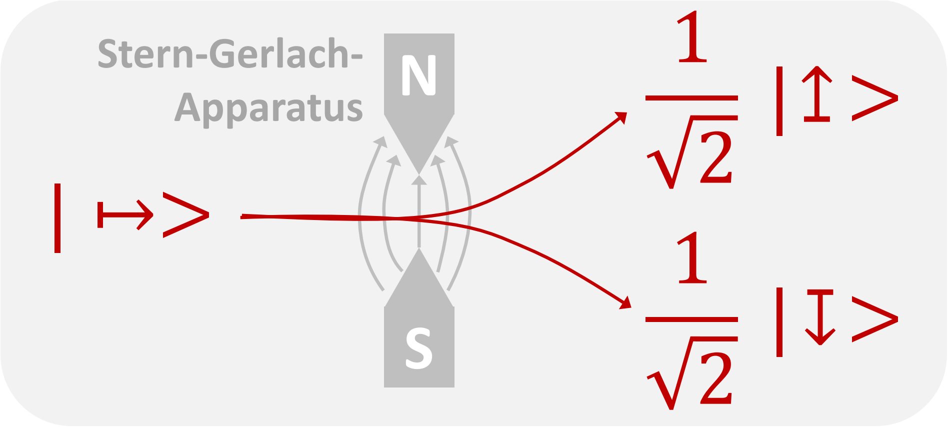
$$\left|\;\right\rangle = \frac{1}{\sqrt{2}}\left|\;\right\rangle + \frac{1}{\sqrt{2}}\left|\;\right\rangle$$

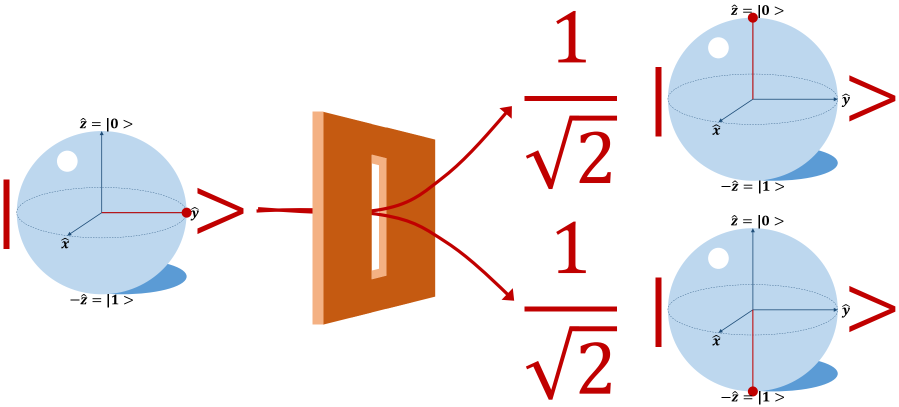
$|1_x > with\ \theta = \frac{\pi}{2}, \phi = \pi:$



$$\left|\;\right\rangle = \frac{1}{\sqrt{2}}\left|\;\right\rangle - \frac{1}{\sqrt{2}}\left|\;\right\rangle$$

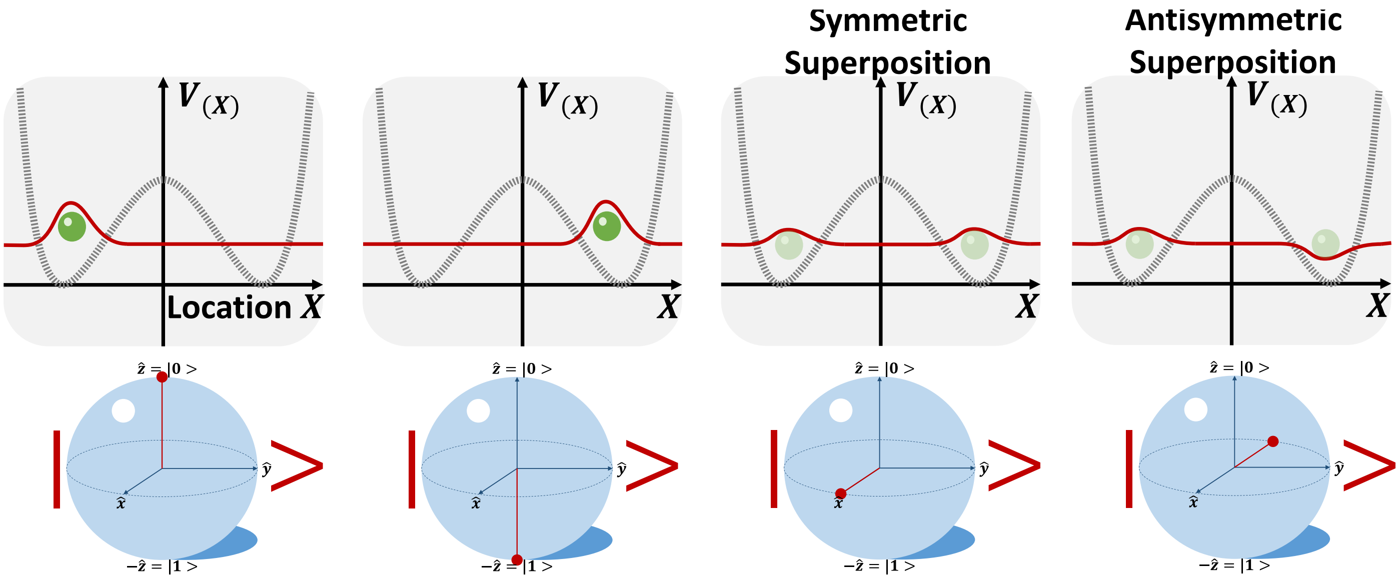
Physical Realizations of a Qubit: Spin of a particle





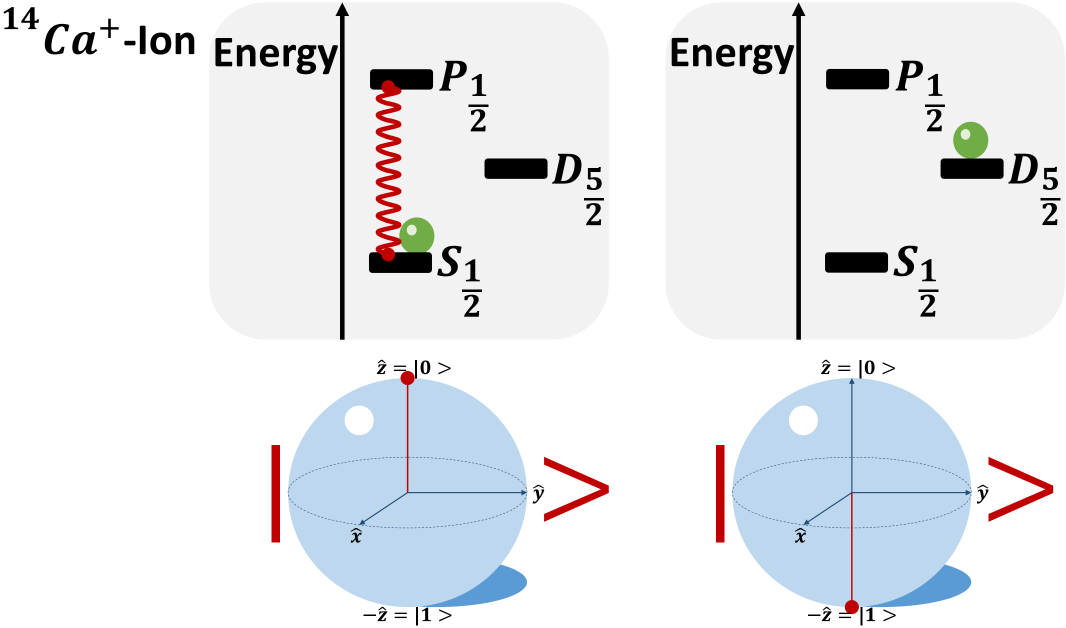
Physical Realization of a Qubit: Localization of Atoms

- Double pot potential:
  - Atom is hold at 2 positions
  - Manipulation of potential barrier: tunneling and hence superposition possible, i.e., manipulation of quantum state
  - Measurement: Where is the atom?



Physical Realizations of a Qubit: Electronic State of an Atom or Ion

- Atom is frozen and fixed in location by a Paul-trap
- Manipulation of quantum states by laser beams of certain frequency and duration
- Measurement: Test via laser beam: if atom is in certain energy level, then moving to helping state via laser beam. Detection of photon when energy level is falling back, otherwise the atom was in the other energy level.



Physical Realizations of a Qubit 1/2

- IBM/Google etc: Superconducting quantum computing (qubit implemented by the state of small superconducting circuits [Josephson junctions])
- Trapped ion quantum computer (qubit implemented by the internal state of trapped ions)
- Neutral atoms in optical lattices (qubit implemented by internal states of neutral atoms trapped in an optical lattice)
- Quantum dot computer, spin-based (e.g. the Loss-DiVincenzo quantum computer) (qubit given by the spin states of trapped electrons)
- Quantum dot computer, spatial-based (qubit given by electron position in double quantum dot)
- Quantum computing using engineered quantum wells, which could in principle enable the construction of quantum computers that operate at room temperature
- Coupled quantum wire (qubit implemented by a pair of quantum wires coupled by a quantum point contact)
- Nuclear magnetic resonance quantum computer (NMRQC) implemented with the nuclear magnetic resonance of molecules in solution, where qubits are provided by nuclear spins within the - dissolved molecule and probed with radio waves
- Solid-state NMR Kane quantum computers (qubit realized by the nuclear spin state of phosphorus donors in silicon)

List adapted from **Wikipedia**

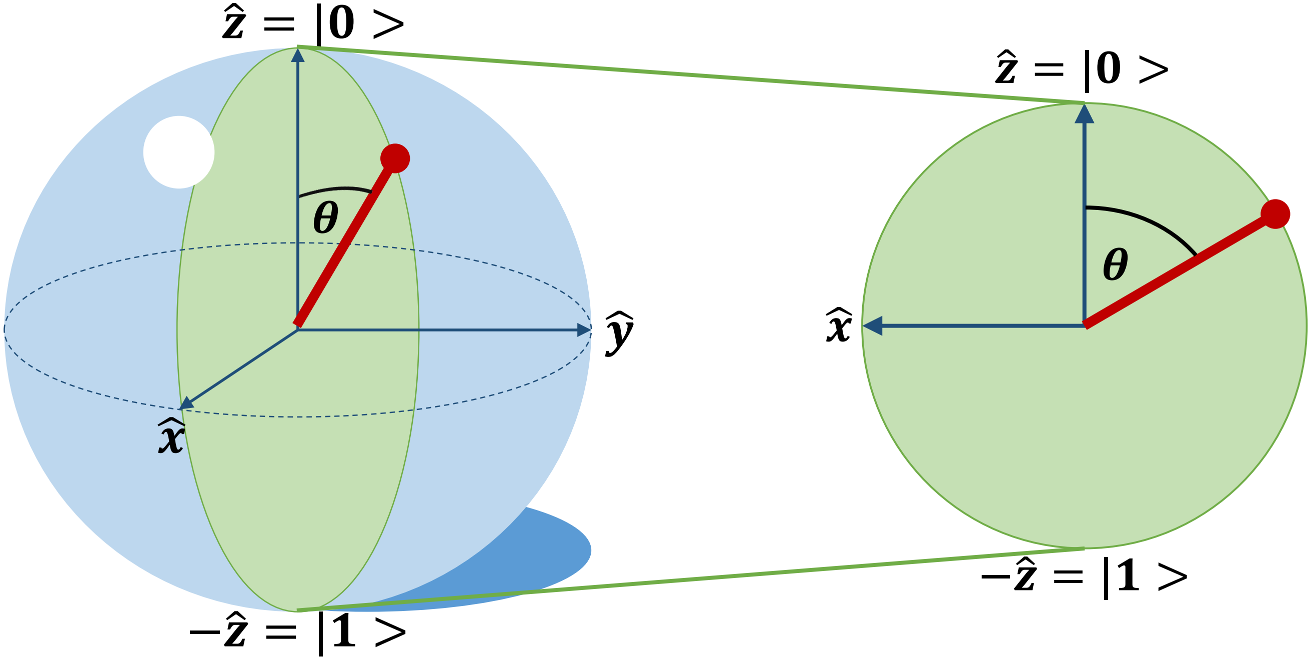
Physical Realizations of a Qubit 2/2

- Electrons-on-helium quantum computers (qubit is the electron spin)
- Cavity quantum electrodynamics (CQED) (qubit provided by the internal state of trapped atoms coupled to high-finesse cavities)
- Molecular magnet (qubit given by spin states)
- Fullerene-based ESR quantum computer (qubit based on the electronic spin of atoms or molecules encased in fullerenes)
- Nonlinear optical quantum computer (qubits realized by processing states of different modes of light through both linear and nonlinear elements)
- Linear optical quantum computer (qubits realized by processing states of different modes of light through linear elements e.g. mirrors, beam splitters and phase shifters)
- E.g. Quantum Brilliance: Diamond-based quantum computer (qubit realized by the electronic or nuclear spin of nitrogen-vacancy centers in diamond)
- Bose-Einstein condensate-based quantum computer
- Transistor-based quantum computer – string quantum computers with entrainment of positive holes using an electrostatic trap
- Rare-earth-metal-ion-doped inorganic crystal based quantum computers (qubit realized by the internal electronic state of dopants in optical fibers)
- Metallic-like carbon nanospheres-based quantum computers

List adapted from **Wikipedia**

More simple representation

- $e^{i\phi} = 1, \theta \in [0, 2\pi]$
  - Avoids problems of complex numbers and complex vector spaces
  - Bloch-sphere is reduced to circle with radius 1



Bra-Ket-Notation (from "bracket", also called Dirac-Notation)

- Ket:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Qubit ψ in superposition: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

- Bra:

$$\langle 0| = [\,1\;0\,]$$
$$\langle 1| = [\,0\;1\,]$$

- Multiplication of Bra with Ket (and vice versa): Matrix multiplication

$$|x\rangle\langle y| = \begin{bmatrix} a \\ b \end{bmatrix} [c\;d] = \begin{bmatrix} a \cdot c & a \cdot d \\ b \cdot c & b \cdot d \end{bmatrix}$$
$$\langle y|x\rangle = [\,c\;d\,] \begin{bmatrix} a \\ b \end{bmatrix} = a \cdot c + b \cdot d$$

Bra-Ket-Notation & multiple Qubits

- Notation for multiple qubits (via tensor product):
  $|\psi\rangle \otimes |\Theta\rangle \equiv |\psi\rangle|\Theta\rangle \equiv |\psi\Theta > \equiv |\psi, \Theta\rangle$
- Examples:

$$|0 > \otimes |1 > = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix} 1\cdot\begin{bmatrix}0\\1\end{bmatrix} \\ 0\cdot\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1\cdot 0\\1\cdot 1\end{bmatrix}\\ \begin{bmatrix}0\cdot 0\\0\cdot 1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ \begin{bmatrix}0\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = |01 >$$

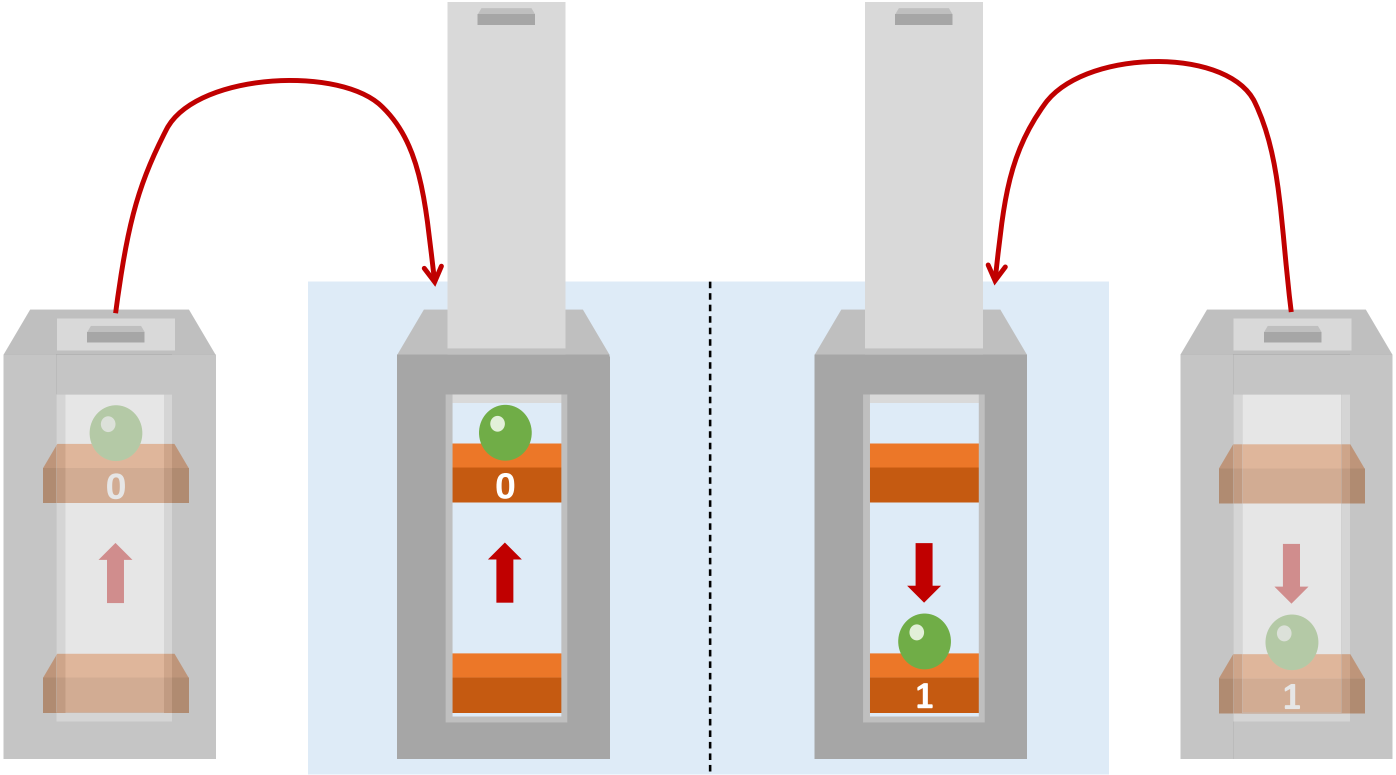
$$(\frac{1}{\sqrt{2}}\cdot|0 > + \frac{1}{\sqrt{2}}\cdot|1 >) \otimes |0 > = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\cdot 1\\ \frac{1}{\sqrt{2}}\cdot 0\\ \frac{1}{\sqrt{2}}\cdot 1\\ \frac{1}{\sqrt{2}}\cdot 0\end{bmatrix} = \frac{1}{\sqrt{2}}\cdot|00 > + \frac{1}{\sqrt{2}}\cdot|10 >$$

$$(\alpha\cdot|0 > + \beta\cdot|1 >) \otimes (\gamma\cdot|0 > + \delta\cdot|1 >) = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \otimes \begin{bmatrix}\gamma\\ \delta\end{bmatrix} = \begin{bmatrix}\alpha\cdot\gamma\\ \alpha\cdot\delta\\ \beta\cdot\gamma\\ \beta\cdot\delta\end{bmatrix} = \alpha\cdot\gamma\cdot|00 > + \alpha\cdot\delta\cdot|01 > + \beta\cdot\gamma\cdot|10 > + \beta\cdot\delta\cdot|11 >$$

- Analogous for more than 2 qubits

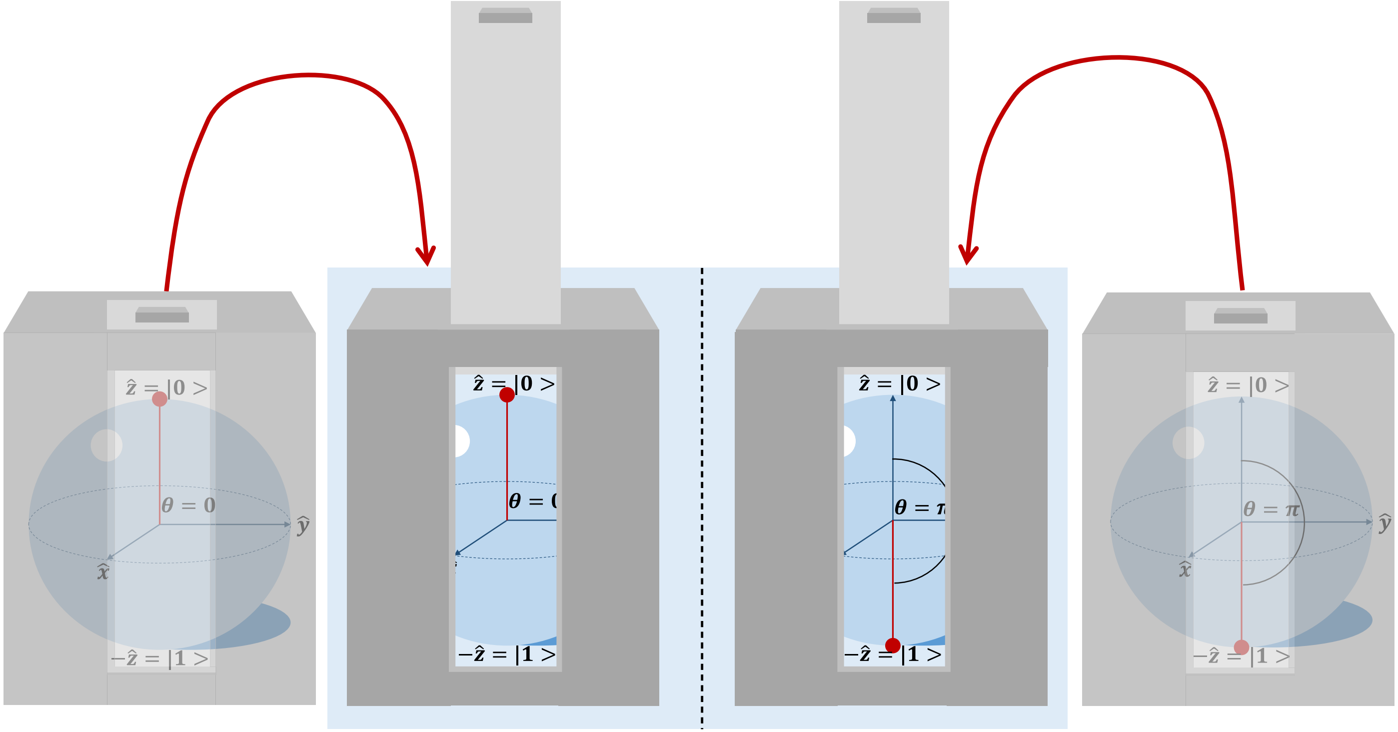
Classical Measurement/Observation

- The state is not destroyed by a measurement/observation in classical systems:



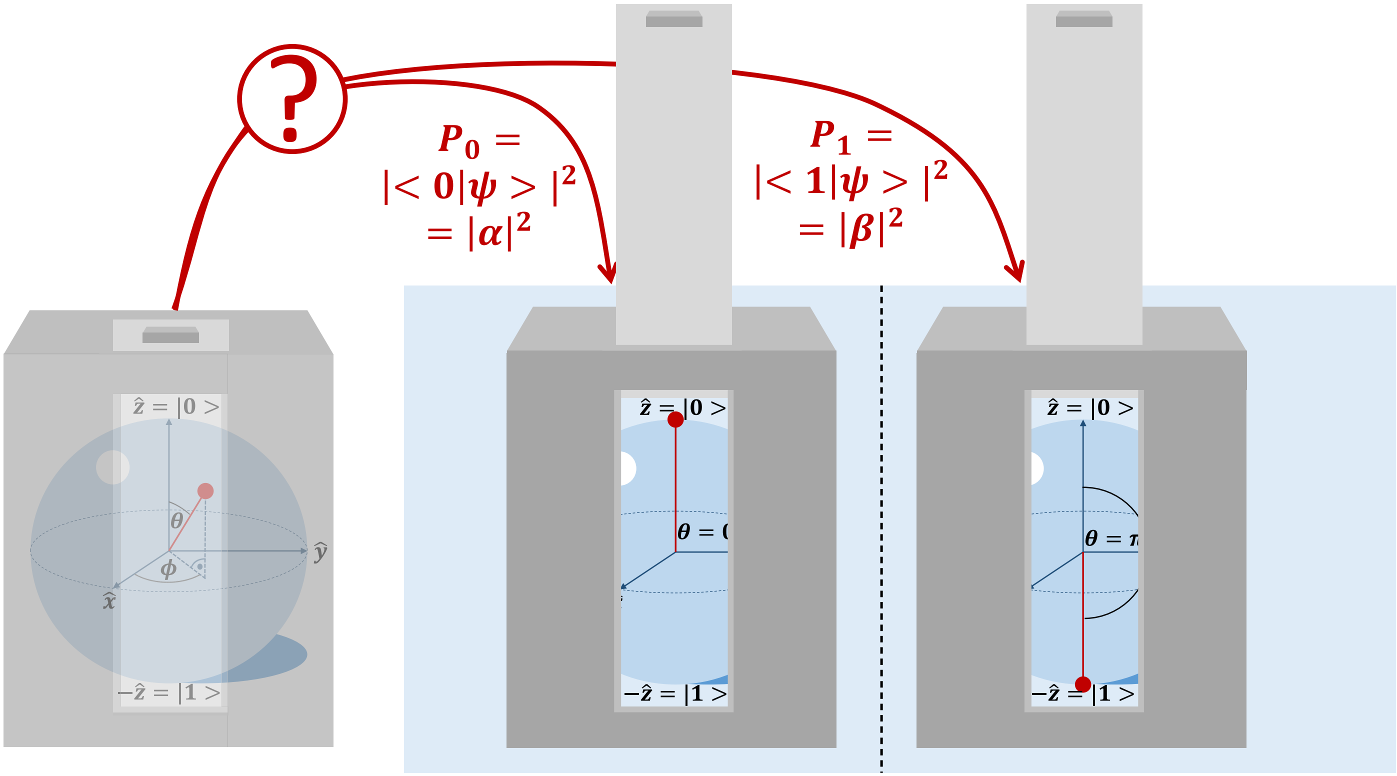
Quantum Measurement/Observation 1/2

- The state is not destroyed by a measurement/observation in quantum mechanical systems for state $|0\rangle$ and $|1\rangle$:



Quantum Measurement/Observation 2/2

- During observation a superposition state collapses to $|0\rangle$ or $|1\rangle$ according to corresponding probabilities:

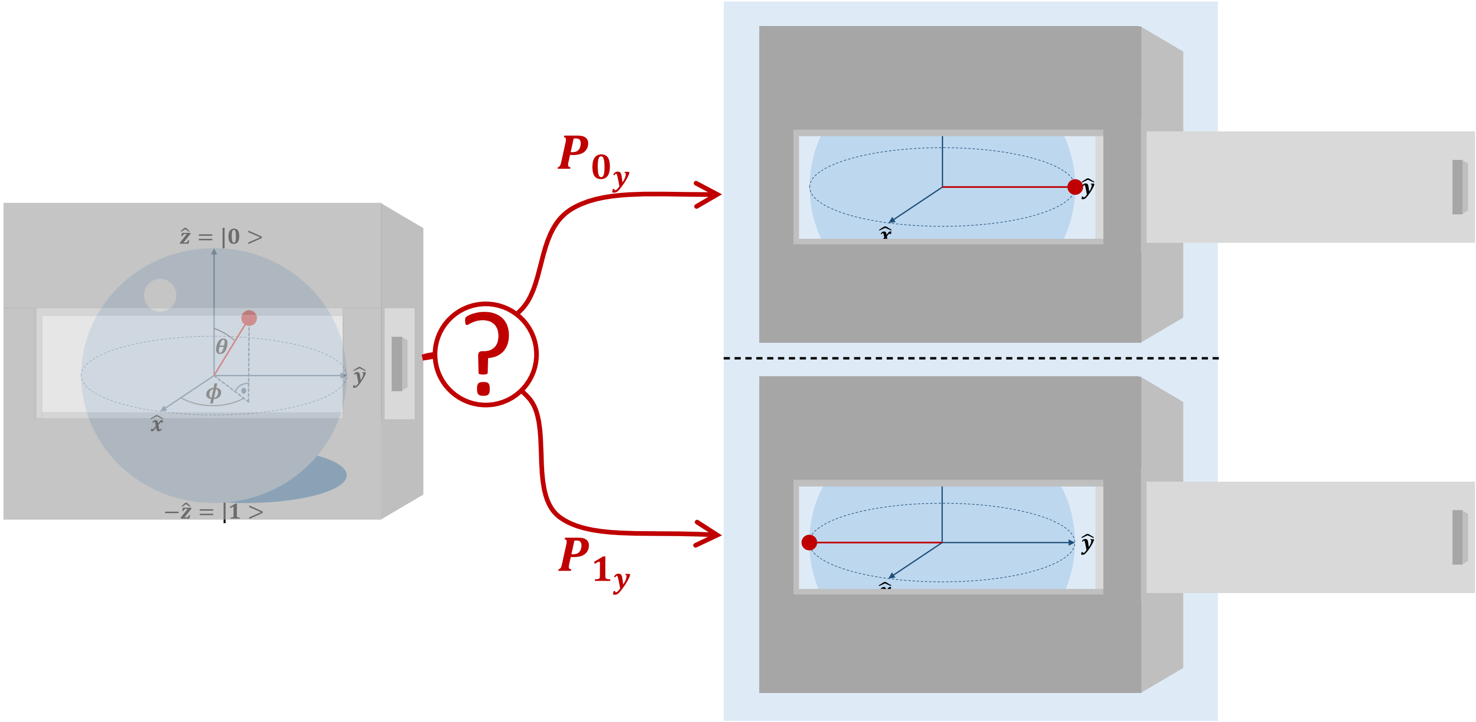


$P_0 = |< 0|\psi >|^2 = |\alpha|^2$

$P_1 = |< 1|\psi >|^2 = |\beta|^2$

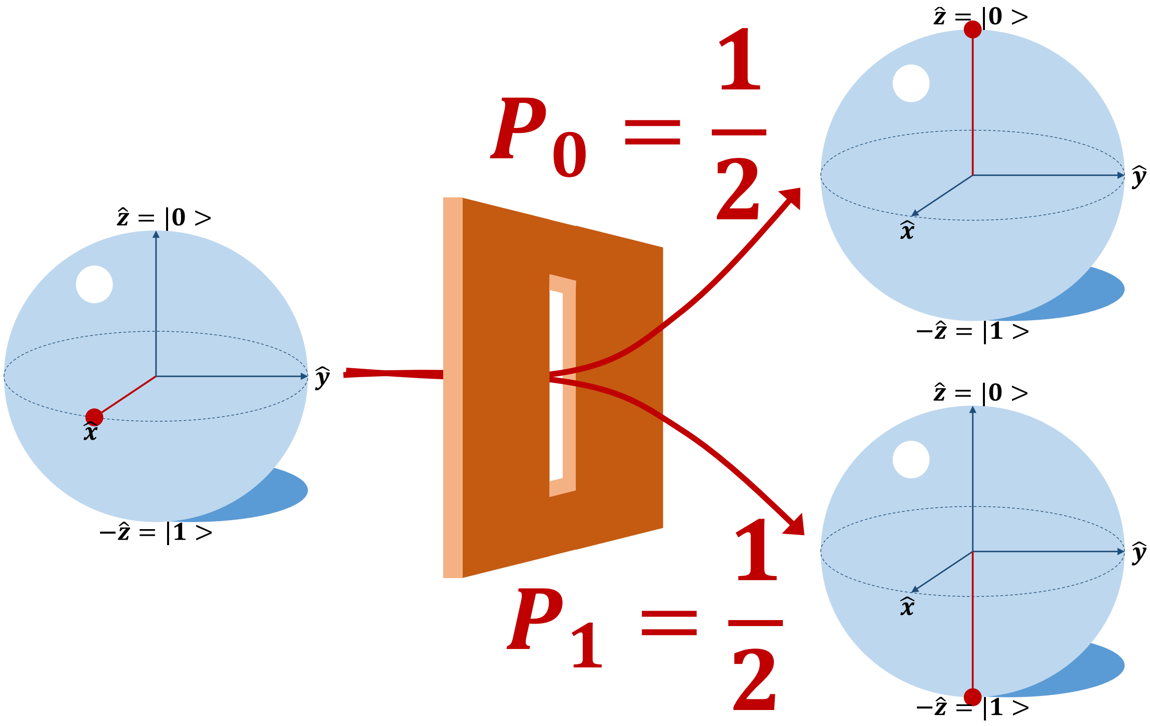
Measurement/Observation along other axis (here y-axis)

- However, observation typically according to z-axis



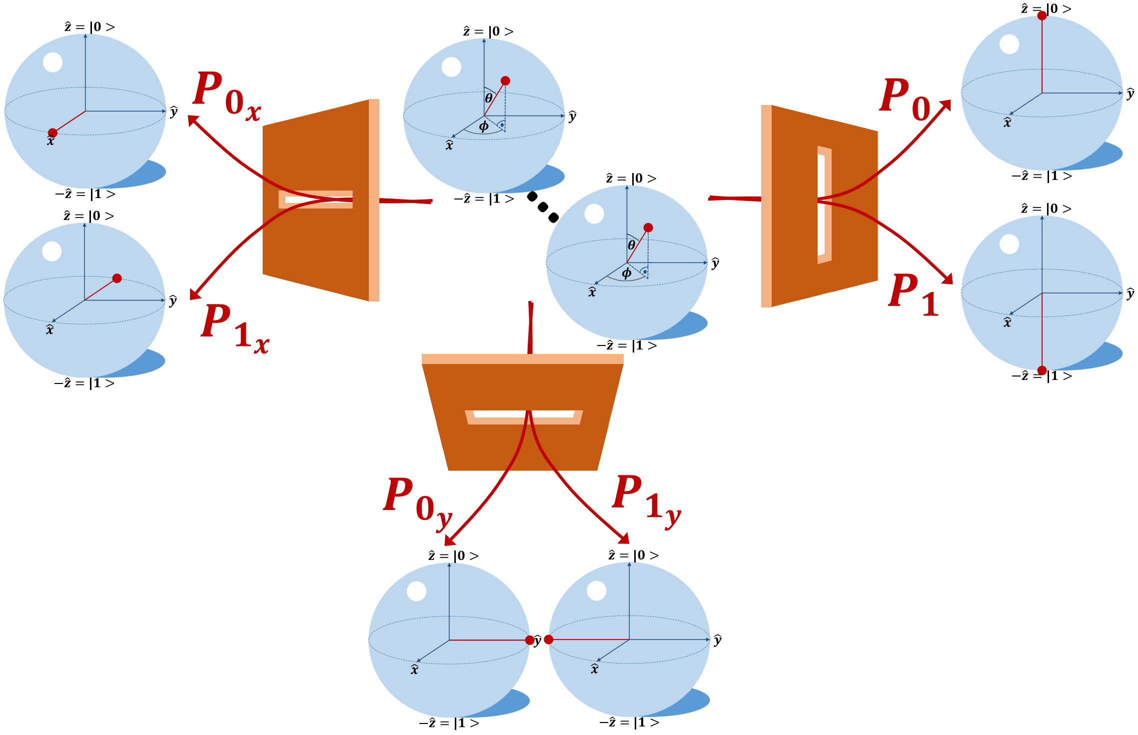
Generator for True Random Numbers

- Commercially available, see e.g.
  - https://www.magiqtech.com/solutions/network-security/
  - https://www.idquantique.com/random-number-generation/



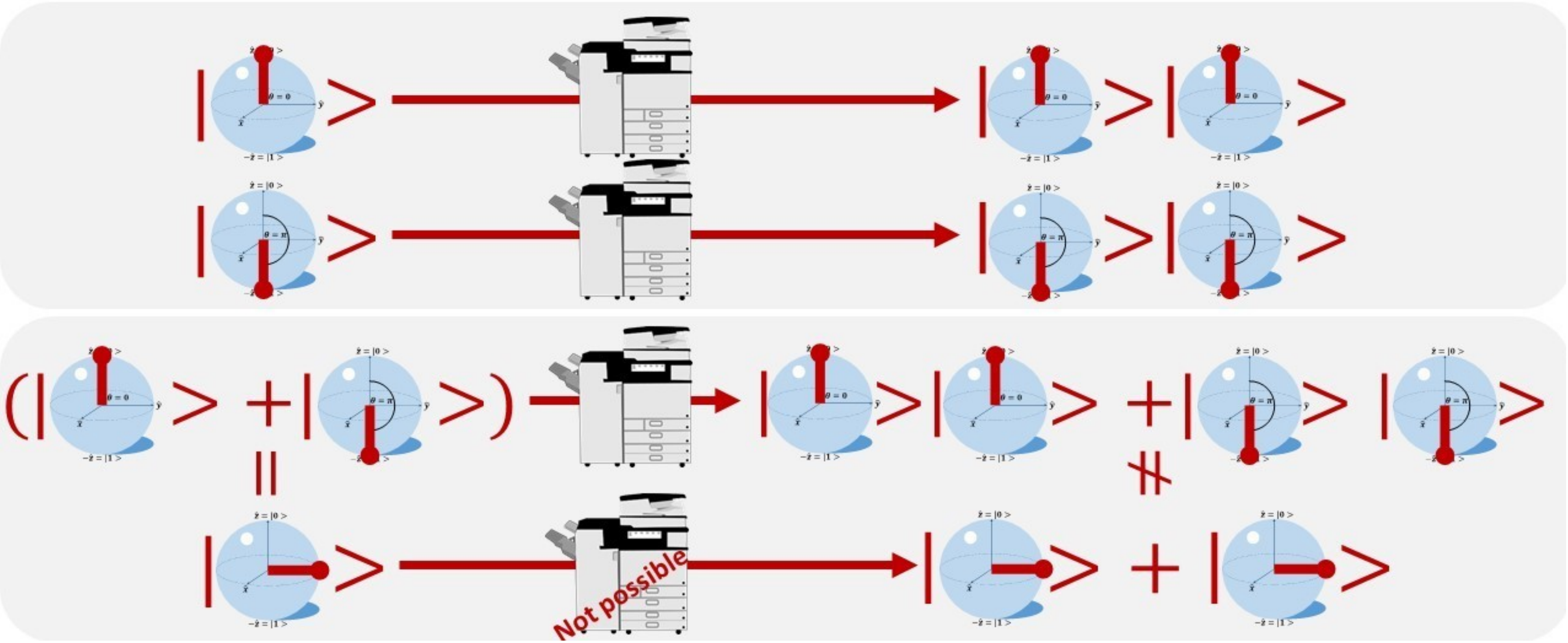
Determining the states (θ, ϕ) of identical prepared Qubits

- After one measurement in one of the axis (x, y, z), the qubit collapses to $|0_a\rangle$ or $|1_a\rangle$ with $a \in \{x, y, z\}$
- As more identical prepared qubits are measured in a axis, as more the measured distribution of $|0_a\rangle$ and $|1_a\rangle$ is getting closer to P_{0_a} and P_{1_a} $\Rightarrow \theta, \phi$ can be determined



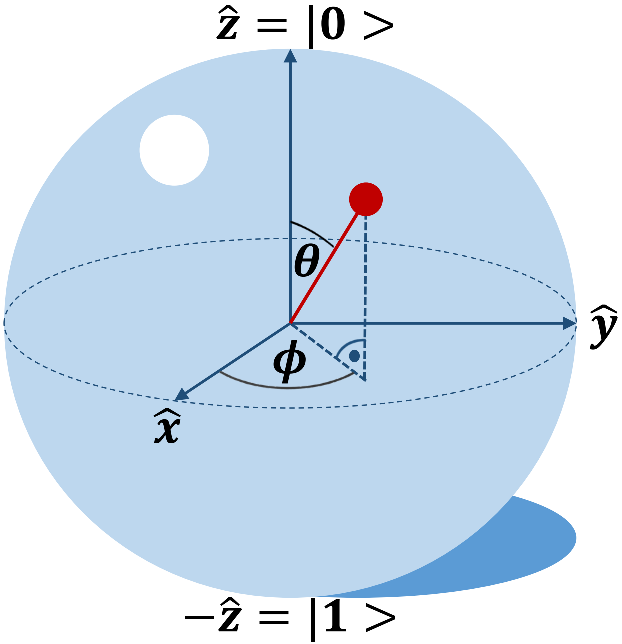
No-Cloning-Theorem of 1 Qubit

- Only not perfect copying possible of information in one of the (x, y, z) axis, other information of superposition gets lost



Operations via Quantum Logic Gates

- quantum logic gates for 1 qubit: often rotation around one axis
  - Relatively general quantum logic gate: rotation around a specified angle θ:



Rotation operator for rotation around x-axis:

$$R_X(\theta) = e^{-iX\frac{\theta}{2}} = \begin{bmatrix} \cos\frac{\theta}{2} & -i\ \sin\frac{\theta}{2} \\ -i\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

e.g. **Pauli$_x$-Gate**:

Rotation Matrix

$$NOT = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Quantum Circuit

$|In\rangle \;-\!\oplus\!-\; |Out\rangle$

Alternatively: $|In\rangle \;-\boxed{X}-\; |Out\rangle$

Table of in- & outputs:

In	Out
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$
$\frac{3\cdot i}{5}\|0\rangle + \frac{4}{5}\|1\rangle$	$\frac{4}{5}\|0\rangle + \frac{3\cdot i}{5}\|1\rangle$

Controlled NOT (CNOT)-Gate

- "*If the control bit is set, then it flips the target bit.*"

Quantum Circuit

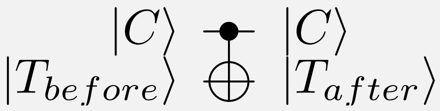


Table of in- & outputs

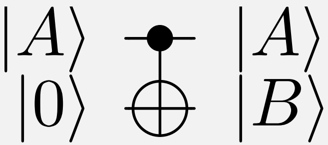
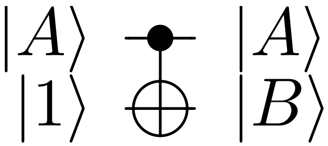
Inputs		Output
Control C	**Target** T_{before}	**Target** T_{after}
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$
$R \cdot \frac{1}{\sqrt{2}}(\|01\rangle + \|11\rangle) = \frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$		

Rotation Matrix R

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- reversible gate: 2 applications of CNOT retrieves the original input
- Classical analog of the CNOT gate is a reversible XOR gate (i.e., they have analogous in- & and outputs for $\{|0\rangle, |1\rangle\}$ inputs)
- $|a, b\rangle \mapsto |a, a \oplus b\rangle$, where $\oplus$ is XOR

Bell States via Entanglement 1/2

<table>
<tr><th>Entanglement</th><th>Quantum Circuit</th><th>Table of in- & outputs</th></tr>
<tr><td>Correlated</td><td>
</td><td><table>
<tr><th>A</th><th>B</th></tr>
<tr><td>$|0\rangle$</td><td>$|0\rangle$</td></tr>
<tr><td>$|1\rangle$</td><td>$|1\rangle$</td></tr>
<tr><td colspan="2">$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$</td></tr>
</table></td></tr>
<tr><td>Anti-Correlated</td><td>
</td><td><table>
<tr><th>A</th><th>B</th></tr>
<tr><td>$|0\rangle$</td><td>$|1\rangle$</td></tr>
<tr><td>$|1\rangle$</td><td>$|0\rangle$</td></tr>
<tr><td colspan="2">$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$</td></tr>
</table></td></tr>
</table>

- Even if the entangled qubits are at different locations, they are still entangled
- Succeeding operations on entangled qubits are possible (without changing the state of the other entangled qubits if the operation is not a measurement)

Bell States via Entanglement 2/2

- Difference to cloning in the sense of no-cloning theorem
  - If one qubit is measured, the other collapses to one of $|0\rangle$ and $|1\rangle$ according to the entanglement and succeeding operations, too.
  - No-cloning theorem refers to that it is not possible to get an independent qubit by copying the superposition of another, which could be independently measured without that the other qubit collapses to $|0\rangle$ or $|1\rangle$
- As computer scientists, I would formulate it in the following way (physicists may forgive me...):
  Some kind of cloning via entanglement and independent succeeding operations is possible, but once one of the (entangled) qubits have been measured, all (entangled) qubits collapse to $|0\rangle$ or $|1\rangle$ with having the same effect as measuring all (entangled) qubits at the same time.

Toffoli-Gate (also called controlled-controlled-not (CCNOT))

- "*If the first two (qu)bits are set, then the Toffoli gate flips the third (qu)bit.*", i.e., it maps (C_1, C_2, T) to $(C_1, C_2, T\ XOR\ (C_1\ \wedge\ C_2))$
- reversible gate

Quantum Circuit

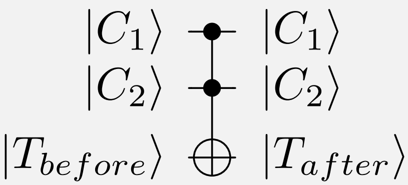


Table of in- & outputs

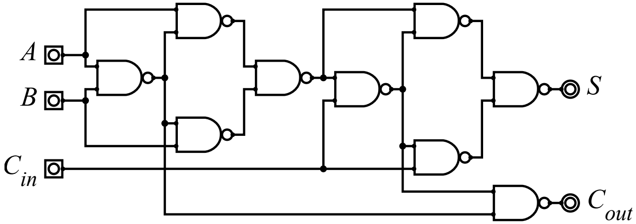
Inputs			Output
C_1	C_2	T_{before}	T_{after}
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$
$\frac{1}{\sqrt{2}} \cdot (\|0\rangle + \|1\rangle)$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$

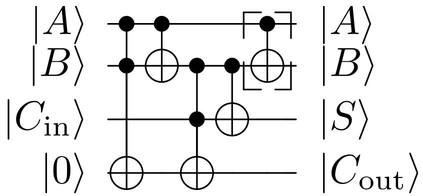
Rotation Matrix R

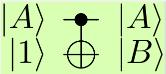
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- Many more quantum logic gates → next lecture unit

Digital versus Quantum Circuits

<table>
<tr><th></th><th>Digital Circuit</th><th>Quantum Circuit</th></tr>
<tr><td>Building Blocks</td><td>Logic Gates</td><td>Quantum Logic Gates</td></tr>
<tr><td>Full Adder Example</td><td>

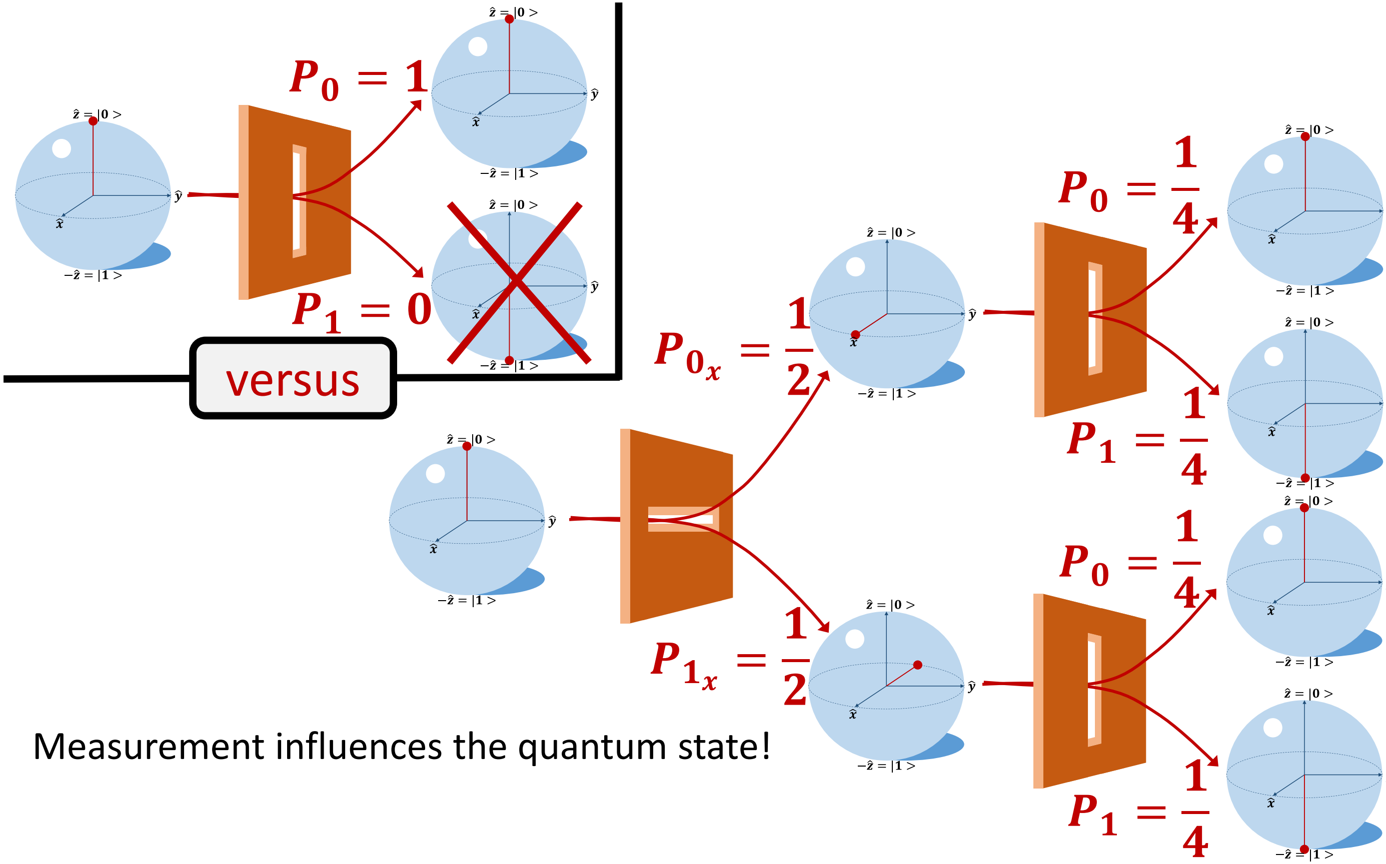
consists of NAND gates</td><td>

consists of Toffoli and CNOT gates[1]</td></tr>
<tr><td>In- and Output Full Adder</td><td>
<table>
<tr><th colspan="3">Inputs</th><th colspan="2">Outputs</th></tr>
<tr><th>A</th><th>B</th><th>C_{in}</th><th>C_{out}</th><th>S</th></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>0</td><td>1</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>0</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>1</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td><td>1</td><td>0</td></tr>
<tr><td>1</td><td>1</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>
</td><td>$|0\rangle$ and $|1\rangle$ as input: Output is $|0\rangle$ and $|1\rangle$ analogous to digital circuit.
Superpositions as input: Superpositions as output with corresponding probabilities for basic quantum states, e.g.:
<table>
<tr><th>A</th><th>B</th><th>C_{in}</th><th>C_{out}</th><th>S</th></tr>
<tr><td>$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$</td><td>$|0\rangle$</td><td>$|0\rangle$</td><td colspan="2">$\mathbf{ABC_{out}S}$:
$\frac{1}{\sqrt{2}}(|0000\rangle + |1001\rangle)$</td></tr>
<tr><td>$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$</td><td>
</td><td>$|0\rangle$</td><td>$|0\rangle$</td><td>$|1\rangle$</td></tr>
</table>
</td></tr>
</table>

---

[1] The dotted square marks a superfluous gate if uncomputation to restore the B output is not required. **[Feynman, 1986]**

Sequence of measurements according to different axis



Measurement influences the quantum state!

PQ Penny Flip - Classical World

The starship Enterprise is facing some immanent—and apparently inescapable—calamity when Q appears on the bridge and offers to help, provided Captain Picard can beat him at penny flipping:

- *Picard is to place a penny head up in a box,*
- *whereupon they will take turns (Q, then Picard, then Q) flipping the penny (or not),*
- *without being able to see it.*
- *Q wins if the penny is head up when they open the box.*

David A. Meyer, Quantum Strategies, Phys. Rev. Lett. 82, 1052, 1999

PQ Penny Flip - Classical World

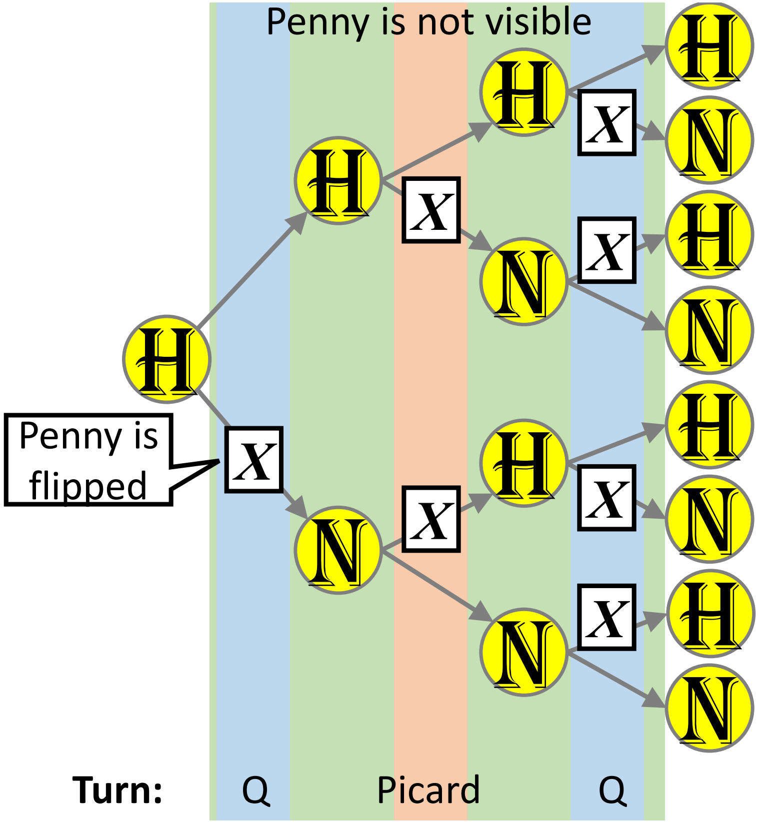
The starship Enterprise is facing some immanent—and apparently inescapable—calamity when Q appears on the bridge and offers to help, provided Captain Picard can beat him at penny flipping:

- *Picard is to place a penny head up in a box,*
- *whereupon they will take turns (Q, then Picard, then Q) flipping the penny (or not),*
- *without being able to see it.*
- *Q wins if the penny is head up when they open the box.*

What is the probability for winning the game?

David A. Meyer, Quantum Strategies, Phys. Rev. Lett. 82, 1052, 1999

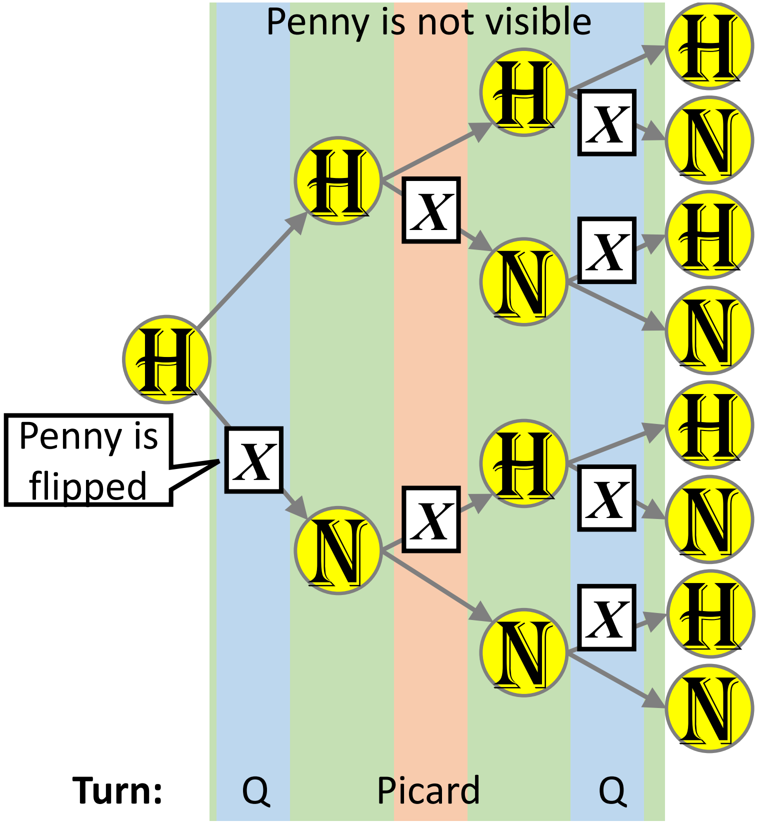
PQ Penny Flip - Classical World



4 ·H(ead)
4 ·N(umber)
⇒ P(H)= 50%
P(T)= 50%

David A. Meyer, Quantum Strategies, Phys. Rev. Lett. 82, 1052, 1999

PQ Penny Flip - Classical World

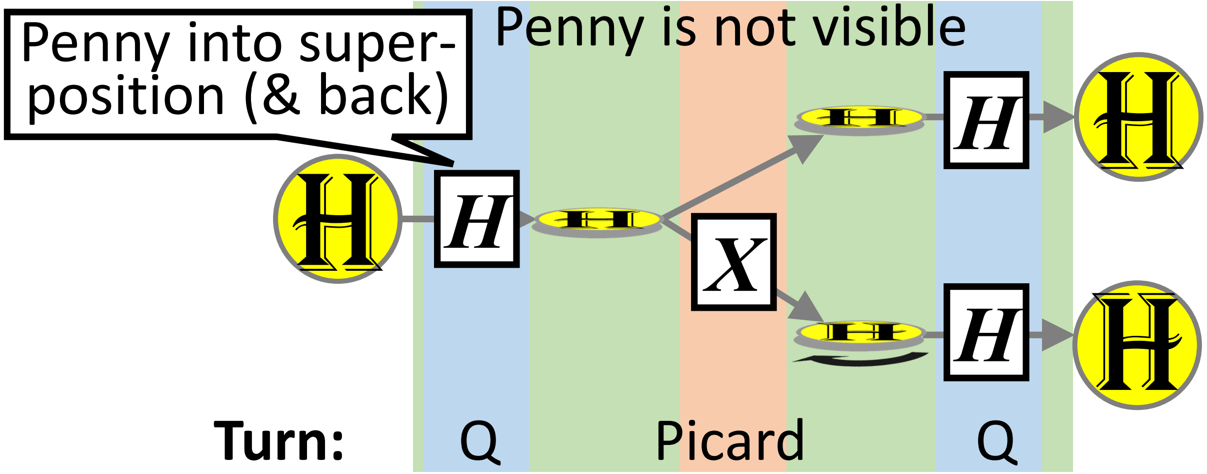


$4 \cdot$H(ead)
$4 \cdot$N(umber)
$\Rightarrow$ P(H)$=$ 50%
P(T)$=$ 50%

What changes if Q is additionally allowed to bring the penny into superposition (and back)?

David A. Meyer, Quantum Strategies, Phys. Rev. Lett. 82, 1052, 1999

PQ Penny Flip - Quantum World



2 ·H(ead)
0 ·N(umber)
⇒ P(H)= 100%
P(T)= 0%

David A. Meyer, Quantum Strategies, Phys. Rev. Lett. 82, 1052, 1999

Summary and Conclusions

- Bloch-Sphere as model for quantum computing
- Basic states and superpositions
- Physical realizations of qubits
- Bra-Ket-Notation
- Measurements/observations
- True Random Generator
- No-cloning principle
- Entanglement
- Quantum computing operations as rotation in bloch sphere
- Sequence of measurements with unexpected results