Lecture

Quantum Computing

(CS5070)

Quantum Logic Gates

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Operations via Quantum Logic Gates

- Operation of a quantum logic gate is defined by
  - its matrix U,
  - which transforms a quantum state $|\psi_1\rangle$ via matrix multiplication
  - into the resulting quantum state $|\psi_2\rangle$:

  $$U|\psi_1\rangle = |\psi_2\rangle$$
- Authors use sometimes different names for the same gates
- Most common gates are introduced on the following slides...
- References, e.g.:

  - Colin P. Williams (2011). Explorations in Quantum Computing. Springer. ISBN 978-1-84628-887-6.
  - Nielsen, Michael A.; Chuang, Isaac (2010). Quantum Computation and Quantum Information. Cambridge: Cambridge University Press. ISBN 978-1-10700-217-3.
  - Yanofsky, Noson S.; Mannucci, Mirco (2013). Quantum computing for computer scientists. Cambridge University Press. ISBN 978-0-521-87996-5.

Identity gate

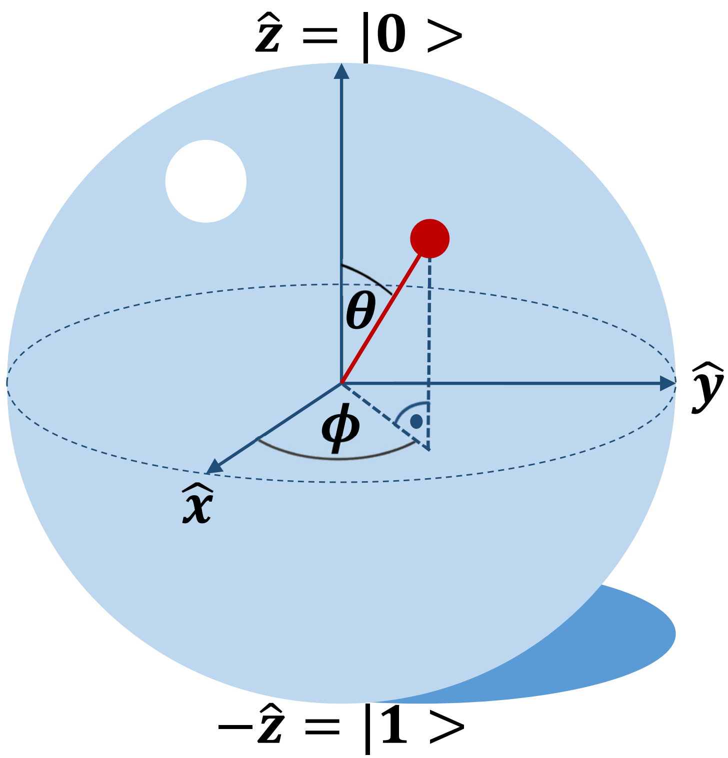
- also known as Pauli I-gate
- defined by the identity matrix
  - for 1 qubit: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - for n qubits: $I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$
- quantum states remain the same:

$$I|\psi_1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \cdot 1 + \beta \cdot 0 \\ \alpha \cdot 0 + \beta \cdot 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = |\psi_2\rangle$$

- I behaves like a NOP (no operation) and can be represented as bare wire in quantum circuits, or not shown at all

Rotation around x-axis

- quantum logic gates for 1 qubit: often rotation around one axis
  - Relatively general quantum logic gate: rotation around a specified angle θ:



Rotation operator for rotation around x-axis:

$$R_X(\theta) = e^{-iX\frac{\theta}{2}} = \begin{bmatrix} cos\frac{\theta}{2} & -i\ sin\frac{\theta}{2} \\ -i\ sin\frac{\theta}{2} & cos\frac{\theta}{2} \end{bmatrix}$$

e.g. **Pauli$_x$-Gate**:

Rotation Matrix

$$X = NOT = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Quantum Circuit

$|In\rangle \oplus |Out\rangle$

Alternatively: $|In\rangle$ $-\boxed{X}-$ $|Out\rangle$

Table of in- & outputs:

In	Out
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$
$\frac{3 \cdot i}{5}\|0\rangle + \frac{4}{5}\|1\rangle$	$\frac{4}{5}\|0\rangle + \frac{3 \cdot i}{5}\|1\rangle$

Pauli$_{x,y,z}$-Gates

- Pauli gates (X,Y,Z): rotation around the x, y and z axes of the Bloch sphere by π radians: $\boldsymbol{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\boldsymbol{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Circuit representation: $|In\rangle$ —[X]— $|Out\rangle$, $|In\rangle$ —[Y]— $|Out\rangle$, $|In\rangle$ —[Z]— $|Out\rangle$
- Pauli matrices are involutory[1] (i.e. square of Pauli matrix is identity matrix): $I^2 = X^2 = Y^2 = Z^2 = -iXYZ = I$
- Pauli matrices are anti-commute[2], e.g., $ZX = iY = -XZ$
- Pauli$_x$-gate (also called bit-flip)
  - quantum equivalent of the NOT gate for classical computers
- Pauli$_y$-gate
  - maps $|0\rangle$ to $i|1\rangle$ and $|1\rangle$ to $-i|0\rangle$
- Pauli$_z$-gate (also called phase-flip)
  - leaves the basis state $|0\rangle$ unchanged and maps $|1\rangle$ to $-|1\rangle$

deutsch: [1] selbst-invers [2] anti-kommutativ

Controlled NOT (CNOT/CX)-Gate

- "*If the control bit is set, then it flips the target bit.*"

Quantum Circuit	Table of in- & outputs	Rotation Matrix R

Quantum Circuit

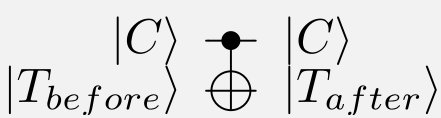


Table of in- & outputs

Inputs		Output
Control C	Target T_{before}	Target T_{after}
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

$$R \cdot \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

Rotation Matrix R

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

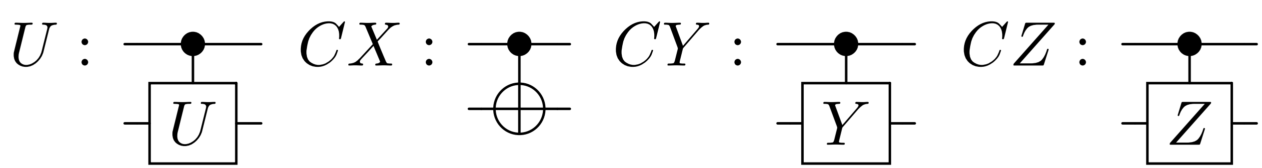
- reversible gate: 2 applications of CNOT retrieves the original input
- Classical analog of the CNOT gate is a reversible XOR gate (i.e., they have analogous in- & and outputs for $\{|0\rangle, |1\rangle\}$ inputs)
- $|a, b\rangle \mapsto |a, a \oplus b\rangle$, where $\oplus$ is XOR

Controlled-U (CU)-Gate

- "If the control bit is set, then the U transformation is applied."

$$U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}, CU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{bmatrix}$$

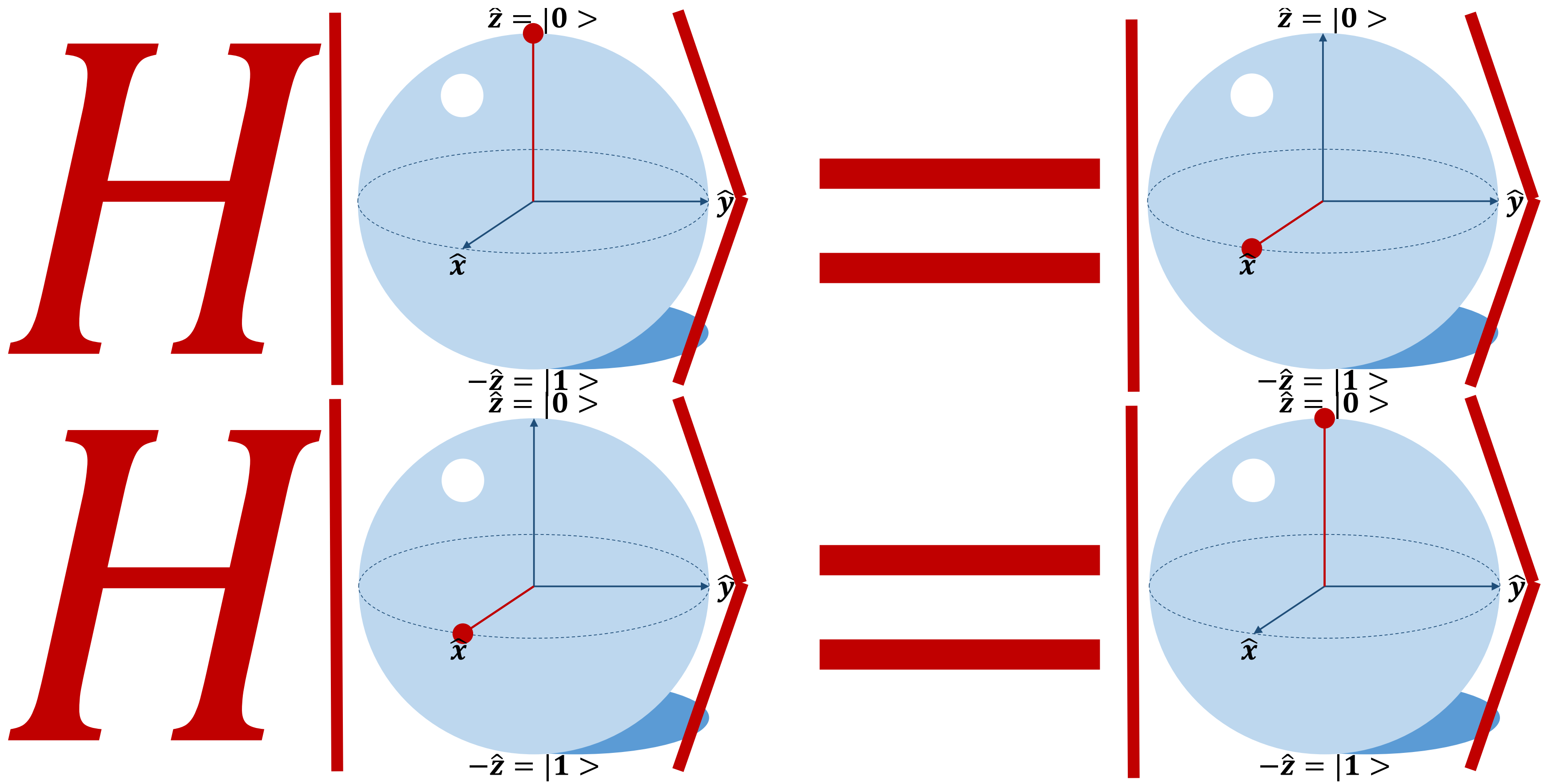
- If $U = X$: "controlled-X" (CX),
  if $U = Y$: "controlled-Y" (CY),
  if $U = Z$: "controlled-Z" (CZ)
- Circuit representations:



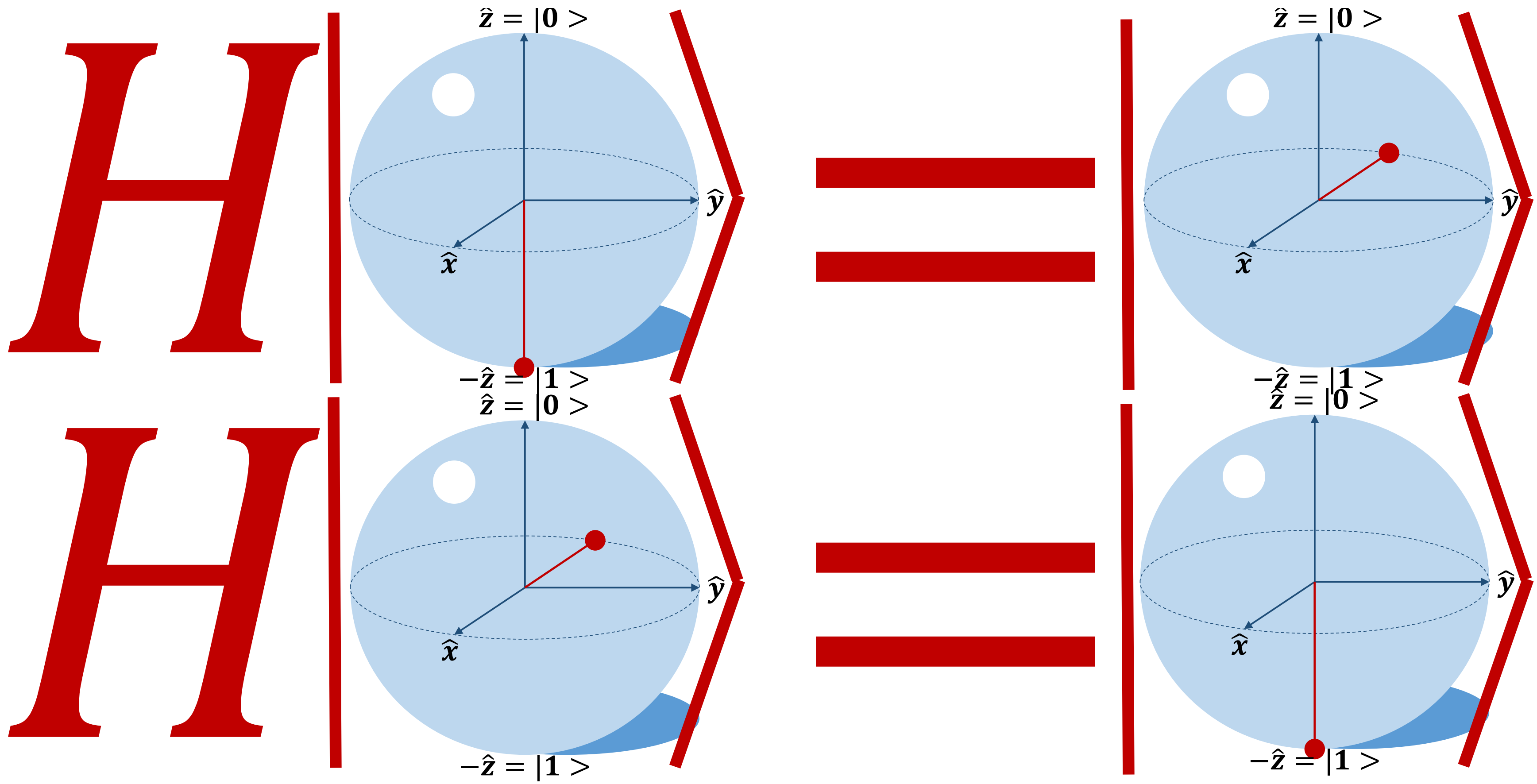
Hadamard (H)-Gate

- unitary transformation that maps qubit operations in z-axis to the x-axis and vice versa
  - represents a rotation of π about the axis $(\hat{x} + \hat{z})/\sqrt{2}$ at the Bloch sphere
  - For example, $HZH = X$, $H\sqrt{X}H = \sqrt{Z} = S$, and $HR_z(\theta)H = R_x(\theta)$
- maps the basis states $|0\rangle \mapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|1\rangle \mapsto \frac{|0\rangle-|1\rangle}{\sqrt{2}}$
  - i.e., it creates a superposition if given a basis state
- $$\boldsymbol{H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$$
- $H^2 = I$, $R_y(\pi/2)Z = H$ and $XR_y(\pi/2) = H$
- Controlled H gate: CU-Gate with $U = H$

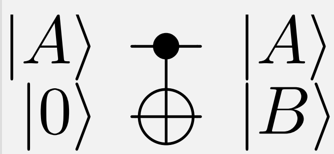
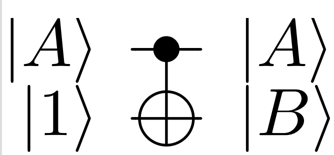
Hadamard transformation on the Bloch Sphere for $|0\rangle$



Hadamard transformation on the Bloch Sphere for $|1\rangle$



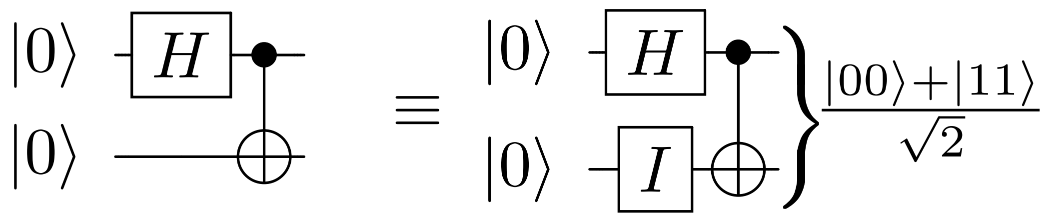
Bell States via Entanglement 1/4

Entanglement	Quantum Circuit	Table of in- & outputs
Correlated		A / B: $\|0\rangle$ / $\|0\rangle$; $\|1\rangle$ / $\|1\rangle$; $\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$
Anti-Correlated		A / B: $\|0\rangle$ / $\|1\rangle$; $\|1\rangle$ / $\|0\rangle$; $\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$

- Even if the entangled qubits are at different locations, they are still entangled (phenomena appears to happen instantaneously ignoring speed of light: open question in physics)
  - As of 2017 experimentally verified for distances of up to 1200 kilometers
- Succeeding operations on entangled qubits do not change the state of the other entangled qubits if the operation is not a measurement

Bell States via Entanglement 2/4

Circuit Representation:



$$H \otimes I = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\ \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & -1\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}$$

$$CNOT(H \otimes I)|00\rangle = \left(\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}\right)\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 0\\ 0\\ 1\end{bmatrix} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

- How many atoms can be entangled?
  - Experiment [C+23] with 2 spatially separated Bose-Einstein condensates, each containing about 700 rubidium atoms
    - Entanglement between the condensates results in strong correlations of their collective spins

Bell States via Entanglement 3/4

- Difference to cloning in the sense of no-cloning theorem
  - If one qubit is measured, the other collapses to one of $|0\rangle$ and $|1\rangle$ according to the entanglement and succeeding operations, too.
  - No-cloning theorem refers to that it is not possible to get an independent qubit by copying the superposition of another, which could be independently measured without that the other qubit collapses to $|0\rangle$ or $|1\rangle$
- As computer scientists, I would formulate it in the following way (physicists may forgive me...):
  Some kind of cloning via entanglement and independent succeeding operations is possible, but once one of the (entangled) qubits have been measured, all (entangled) qubits collapse to $|0\rangle$ or $|1\rangle$ with having the same effect as measuring all (entangled) qubits at the same time.

Bell States via Entanglement 4/4

- Verifying entanglement of quantum states:
  None of the Bell states can be broken into (tensor) products!
- e.g., for 2 qubits: $\begin{bmatrix} x \\ y \end{bmatrix} \otimes \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} x \cdot w \\ x \cdot z \\ y \cdot w \\ y \cdot z \end{bmatrix}$
  - $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow y = z = 0, x \cdot w = 1$, i.e., **no entanglement**!
  - $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow (x = 0 \vee w = 0) \wedge x \neq 0 \wedge w \neq 0$ ↯, i.e., **entanglement**!

Phase shift gates

- map the basis states $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto e^{i\varphi}|1\rangle$
- probability of measuring a $|0\rangle$ or $|1\rangle$ is unchanged
- modify the phase of the quantum state. This is equivalent to tracing a horizontal circle (a line of latitude) on the Bloch sphere* by φ radians
- $P(\varphi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}$, where φ is the phase shift with the period 2π
- $P(\pi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} =: Z$

  $P(\frac{\pi}{2}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \sqrt{Z} =: S$ (although sometimes used for SWAP)

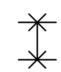
  $P(\frac{\pi}{4}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} = \sqrt[4]{Z} = \sqrt[2]{S} =: T$
- Controlled phase shift gates: CU-Gate with $U = P(\varphi)$

* deutsch: Dies entspricht dem Zeichnen eines horizontalen Kreises (einer Breitenlinie) auf der Blochkugel

Swap-Gate ($SWAP$)

- swaps two qubits:

$$SWAP := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Circuit Representation: 

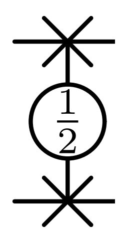
$$SWAP(\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle) = SWAP \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \\ \gamma \\ \beta \\ \delta \end{bmatrix} = \alpha|00\rangle + \gamma|01\rangle + \beta|10\rangle + \delta|11\rangle$$

Square root of Swap-Gate ($\sqrt{SWAP}$)

- arises naturally in systems that exploit exchange interaction*
- universal gate such that any many-qubit gate can be constructed from only $\sqrt{SWAP}$ and single qubit gates
- not maximally entangling (i.e., more than one application of it is required to produce a Bell state)
- performs half-way of a two-qubit swap:

$$\sqrt{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1+i) & \frac{1}{2}(1-i) & 0 \\ 0 & \frac{1}{2}(1-i) & \frac{1}{2}(1+i) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Circuit Representation:



Toffoli-Gate (controlled-controlled-not (CCNOT)/Deutsch gate $D(\frac{\pi}{2})$)

- "*If the first two (qu)bits are set, then the Toffoli gate flips the third (qu)bit.*", i.e., it maps (C_1, C_2, T) to $(C_1, C_2, T\ XOR\ (C_1 \wedge C_2))$
- reversible gate
- universal: for classical computation and for quantum computation when combined with the single qubit Hadamard gate

Quantum Circuit

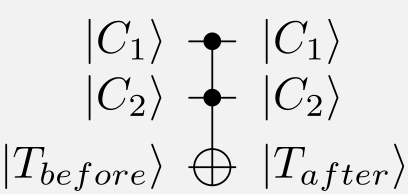


Table of in- & outputs

C_1	C_2	T_{before}	T_{after}
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$
$\frac{1}{\sqrt{2}} \cdot (\|0\rangle + \|1\rangle)$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$

Rotation Matrix R

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Fredkin-Gate ($CSWAP/CS$)

- 3-bit gate that performs a controlled swap
- universal for classical computation

Quantum Circuit

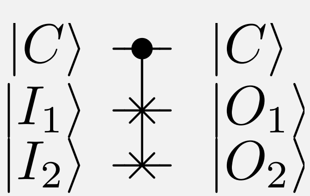


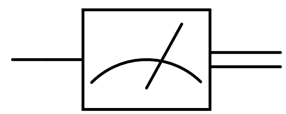
Table of in- & outputs

C	I_1	I_2	O_1	O_2
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\frac{1}{\sqrt{2}} \cdot (\|0\rangle + \|1\rangle)$	$\frac{1}{\sqrt{2}} \cdot (\|0\rangle + \|1\rangle)$	$\|0\rangle$

Rotation Matrix R

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

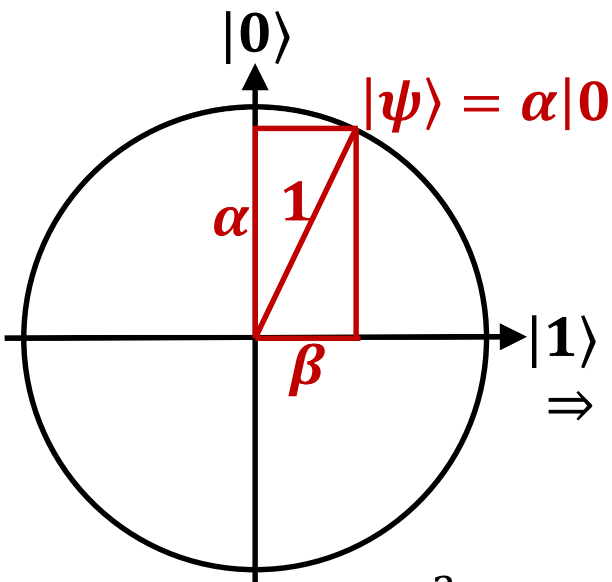
Measurement (also called observation)

Input		Circuit Represen-tation	Output	
Information Unit	Line		Line	Information Unit
Qubit	Single		Double	Classical Bit

- irreversible and therefore not a quantum gate, because it assigns the observed quantum state to a single value

Measurement - Probabilities for basis state collapses

1 Qubit



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Unit Sphere in $\mathbb{C}^2$

$$\Rightarrow |\alpha|^2 + |\beta|^2 = 1$$

Measuring:

$$P(|0\rangle) = |\alpha|^2$$
$$P(|1\rangle) = |\beta|^2$$

n Qubits: Generalization to $\mathbb{C}^{2^n}$

$$|\psi\rangle = \sum_{x=0}^{2^n-1} \alpha_x |x\rangle \in \mathbb{C}^{2^n}$$

$$\Rightarrow \sum_{x=0}^{2^n-1} |\alpha_x|^2 = 1$$

Measuring:

$$P(|x\rangle) = |\alpha_x|^2$$

Set of Universal Quantum Gates

- is any set of gates to which any operation can be reduced
  - In other words: any other unitary operation can be expressed as a finite sequence of gates from this set
- Examples of universal quantum gates sets
  - Rotation operators $R_x(\theta), R_y(\theta), R_z(\theta)$, the phase shift gate $P(\varphi), CNOT$
  - $CNOT, H, S$ and T gates
  - Two-gate set of universal quantum gates: Toffoli and Hadamard gates
  - ...

Unitary Inversion † of Gates 1/2

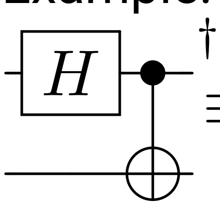
- All quantum logical gates are reversible $\Rightarrow$ any composition of multiple gates is also reversible
- Series and parallel combinations of unitary matrices are also unitary matrices $\Rightarrow$ Inversing all algorithms and functions containing only gates is possible
  - Inversion not possible for initialization, measurement, I/O and spontaneous decoherence
- U is a unitary matrix $\Rightarrow U^{\dagger}U = UU^{\dagger} = I$ and $U^{\dagger} = U^{-1}$
  - The dagger $\dagger$ denotes the conjugate transpose (also called the Hermitian adjoint, deutsch: adjungierte/hermitesch transponierte/transponiert-konjugierte Matrix)
- used in uncomputation* for cleaning up temporary effects on ancilla (i.e., auxiliary) bits so that they can be re-used

* S. Aaronson, D. Grier, L. Schaeffer. The Classification of Reversible Bit Operations. arXiv:1504.05155, 2015

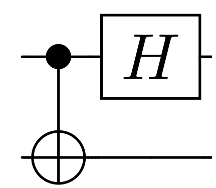
Unitary Inversion † of Gates 2/2

- Hermitian (also called self-adjoint operators[1]): Gates that are their own unitary inverses like Hadamard (H) and the Pauli gates (I, X, Y, Z)
- Skew-Hermitian[2] (also called adjoint operators): Gates that are not their own unitary inverses in general like the phase shift ($S, T, P, CPHASE$)
- $(UV)^\dagger = V^\dagger U^\dagger, (A_1 \cdots A_m)^\dagger = A_m^\dagger \cdots A_1^\dagger$
- $(U \otimes V)^\dagger = U^\dagger \otimes V^\dagger, (A_1 \otimes \cdots \otimes A_m)^\dagger = A_1^\dagger \otimes \cdots \otimes A_m^\dagger$

Example:



$$\equiv (CNOT(H \otimes I))^\dagger = (H^\dagger \otimes I^\dagger)CNOT^\dagger = (H \otimes I)CNOT \equiv$$



Circuit composition - Serially Wired Gates

- When gate B is put after gate A in a series circuit, then the effect of the two gates can be described as a single gate $C = B \cdot A$
- Example:

$$|\psi_1\rangle \; -\boxed{Y}-\boxed{X}- \; |\psi_2\rangle$$

$$\equiv$$

$$|\psi_1\rangle \; -\boxed{C = X \cdot Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = iZ}- \; |\psi_2\rangle = XY\,|\psi_1\rangle = C\,|\psi_1\rangle$$

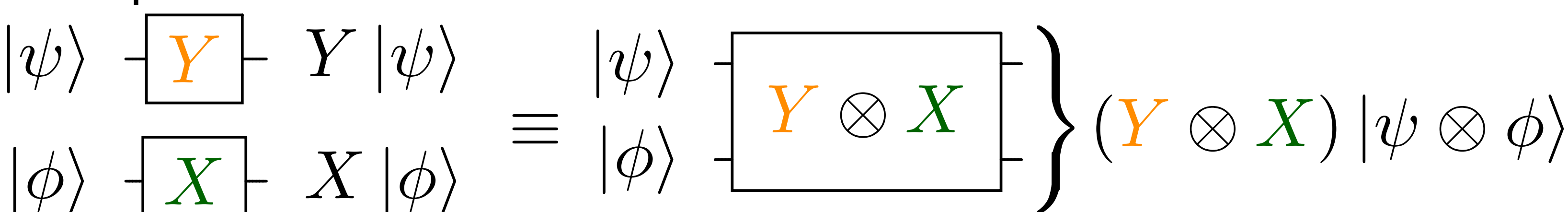
Exponents of Quantum Gates

- All quantum gates are unitary matrices
- Positive integer exponents are equivalent to sequences of serially wired gates
  - e.g., $X^3 = X \cdot X \cdot X$
- Real exponents is a generalization of the series circuit
  - All real exponents of unitary matrices are also unitary matrices/quantum gates
  - e.g., X^π and $\sqrt{X} = X^{\frac{1}{2}}$ are valid quantum gates
- $U^0 = I$ for any unitary matrix U
- $U^{-n} = (U^n)^\dagger$
  - e.g., $T^{-1} = T^\dagger$ and $T^{-2} = (T^2)^\dagger = S^\dagger$

Circuit composition - Parallel Gates

- The tensor product (or Kronecker product) of two quantum gates is the gate that is equal to the two gates in parallel

  Example:



$$C = Y \otimes X = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & -i \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ i \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

Circuit composition - Parallel Hadamard

- Parallel application of Hadamard gates on n qubits (all in basis state $|0\rangle$):

$$\underbrace{H \otimes H \otimes \cdots \otimes H}_{n \text{ times}} = \bigotimes_1^n H = H^{\otimes n} = H_n$$

$$\bigotimes_1^n (H|0\rangle) = \left(\bigotimes_1^n H\right)\left(\bigotimes_1^n |0\rangle\right) = \frac{1}{\sqrt{2^n}}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = \frac{|0\rangle+|1\rangle+\cdots+|2^n-1\rangle}{\sqrt{2^n}} = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle$$

- This state is a uniform superposition*:
  - If not measured, it is a quantum state with equal probability amplitude $\frac{1}{\sqrt{2^n}}$ for each of its possible states
  - Measuring this state results in a random number between $|0\rangle$ and $|2^n-1\rangle$
  - Often first step of quantum computing algorithms like Grover's search

* deutsch: gleichmäßige Überlagerung

Computational Complexity of Simulating Quantum Computers

- Time complexity for multiplying two $k \times k$-matrices is at least $\Omega(k^2 \log k)$ on a classical machine*
- For n-qubits-gate: $k = 2^n$
  $\Rightarrow$ intractable to simulate large entangled quantum systems using classical computers
- Efficient simulation of subsets of the gates, such as the Clifford gates, or combinations of $X, CNOT$, Toffoli
- State vector of a quantum register is 2^n complex entries
  $\Rightarrow$ Storing the probability amplitudes as a list of floating point values is not tractable for large n

* R. Raz, On the complexity of matrix product. STOC, 2002

More Quantum Logic Gates...

- Ising coupling gates
  - implemented natively in some trapped-ion quantum computers
    - Debnath et al. Demonstration of a small programmable quantum computer with atomic qubits. Nature 536, 63–66,2016
- Imaginary swap ($iSWAP$) and it root version $\sqrt{iSWAP}$
  - for systems with Ising like interactions
    - S.E. Rasmussen, N.T. Zinner. Simple implementation of high fidelity controlled- i swap gates and quantum circuit exponentiation of non-Hermitian gates. Physical Review Research. 2(3), 2020
    - N. Schuch, J. Siewert. Natural two-qubit gate for quantum computation using the XY interaction. Physical Review A. 67(3), 2003
    - P.-L. Dallaire-Demers, F.K. Wilhelm. Quantum gates and architecture for the quantum simulation of the Fermi-Hubbard model. Physical Review A. 94 (6), 2016
- Deutsch gate (named after physicist David Deutsch)
  - some proposals to realize a Deutsch gate with dipole-dipole interaction in neutral atoms
    - X.-F. Shi. Deutsch, Toffoli, and cnot Gates via Rydberg Blockade of Neutral Atoms. Physical Review Applied. 9(5), 2018
- ...

COVID-19 Variant of the El Farol Bar Problem for simplicity reasons for 4 friends

- 4 friends want to go to a restaurant for lunch.
- However, because of COVID-19 confinements, only 3 guests are allowed to sit at a table of a restaurant.
- If all friends go to this restaurant, then no one will go inside because of solidarity, because one of them would stay hungry if 3 of them go into the restaurant.
- If 3 go to restaurant, then these 3 have most fun compared to only 1, 2 or none of them go to restaurant.
- The friends do not want to decide which of them goes to the restaurant. Therefore, they independently decide at home without communication whether or not to go to the restaurant.

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After some trials the friends are not satisfied with the procedure. One of the friends studied quantum information theory and proposes: **"Let's decide it by a quantum computer!"**

COVID-19 Variant of the El Farol Bar Problem for simplicity reasons for 4 friends

- k-th friend gets the k-th qubit. If the k-th qubit is set, then the k-th friend stays at home and the other friends go to restaurant
- Hence the quantum circuit should deliver one of $|0001\rangle, |0010\rangle, |0100\rangle, |1000\rangle$ with equal probability $\rightsquigarrow$ (General) W-State (for n qubits)

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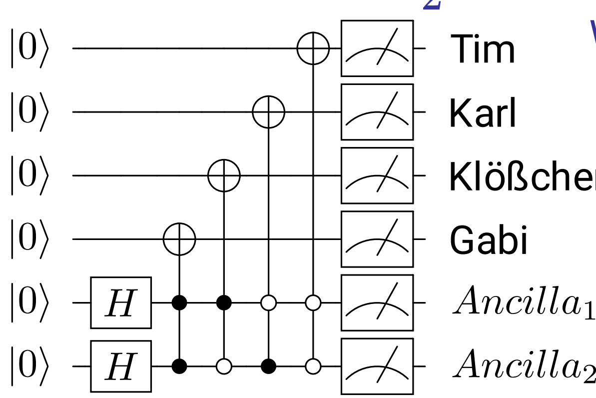
With ancilla:



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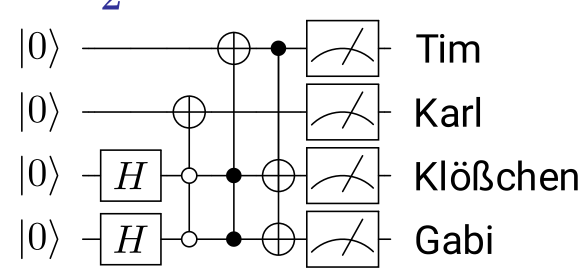
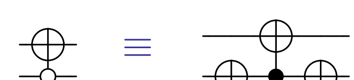


Legend:



Summary and Conclusions

- Quantum Logic Gates
  - Rotation: Pauli (I, X, Y, Z), $R_x, R_y, R_z, H, P(\varphi)$
  - $SWAP, \sqrt{SWAP}$
  - Controlled Gates: $CX, CU, CCNOT, CSWAP$
  - Measurement
  - ...
- Bell States via Entanglement
- Universal Quantum Logic Gates
- Unitary Inversion
- Circuit
  - Composition
  - Simulation