Lecture

Quantum Computing

(CS5070)

Quantum Logic Gates

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Operations via Quantum Logic Gates

- Operation of a quantum logic gate is defined by
  - its matrix U,
  - which transforms a quantum state $|\psi_1\rangle$ via matrix multiplication
  - into the resulting quantum state $|\psi_2\rangle$:

    $U|\psi_1\rangle = |\psi_2\rangle$
- Authors use sometimes different names for the same gates
- Most common gates are introduced on the following slides...
- References, e.g.:
  - Colin P. Williams (2011). Explorations in Quantum Computing. Springer. ISBN 978-1-84628-887-6.
  - Nielsen, Michael A.; Chuang, Isaac (2010). Quantum Computation and Quantum Information. Cambridge: Cambridge University Press. ISBN 978-1-10700-217-3.
  - Yanofsky, Noson S.; Mannucci, Mirco (2013). Quantum computing for computer scientists. Cambridge University Press. ISBN 978-0-521-87996-5.

Identity gate

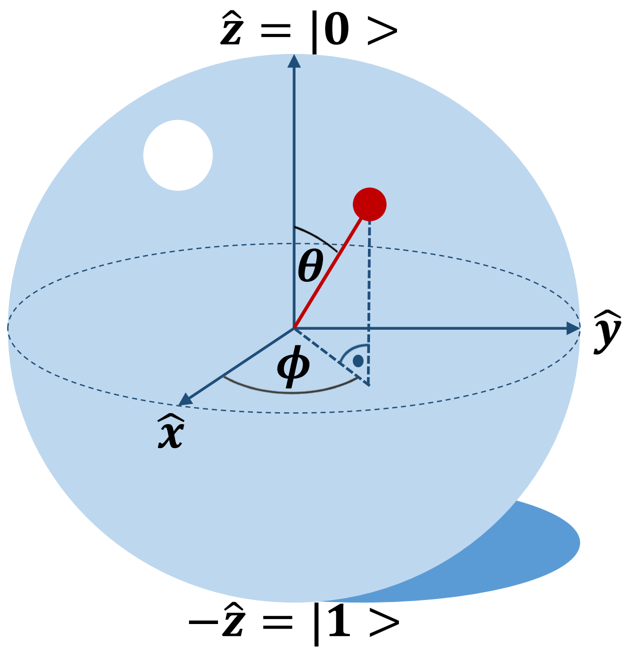
- also known as Pauli I-gate
- defined by the identity matrix
  - for 1 qubit: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - for n qubits: $I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$
- quantum states remain the same:

$$I|\psi_1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \cdot 1 + \beta \cdot 0 \\ \alpha \cdot 0 + \beta \cdot 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = |\psi_2\rangle$$

- I behaves like a NOP (no operation) and can be represented as bare wire in quantum circuits, or not shown at all

Rotation around x-axis

- quantum logic gates for 1 qubit: often rotation around one axis
  - Relatively general quantum logic gate: rotation around a specified angle θ:



Rotation operator for rotation around x-axis:

$$R_X(\theta) = e^{-iX\frac{\theta}{2}} = \begin{bmatrix} \cos\frac{\theta}{2} & -i\ \sin\frac{\theta}{2} \\ -i\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

e.g. **Pauli$_x$-Gate**:

Rotation Matrix	$X = NOT = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Quantum Circuit	$\|In\rangle \oplus \|Out\rangle$ Alternatively: $\|In\rangle\ \boxed{X}\ \|Out\rangle$
Table of in- & outputs:	see table below

In	Out
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$
$\frac{3 \cdot i}{5}\|0\rangle + \frac{4}{5}\|1\rangle$	$\frac{4}{5}\|0\rangle + \frac{3 \cdot i}{5}\|1\rangle$

Pauli$_{x,y,z}$-Gates

- Pauli gates (X,Y,Z): rotation around the x, y and z axes of the Bloch sphere by π radians: $\boldsymbol{X} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \boldsymbol{Y} = \begin{bmatrix} \mathbf{0} & -\boldsymbol{i} \\ \boldsymbol{i} & \mathbf{0} \end{bmatrix}, \boldsymbol{Z} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix}$
- Circuit representation: $|In\rangle$ $-\boxed{X}-$ $|Out\rangle$, $|In\rangle$ $-\boxed{Y}-$ $|Out\rangle$, $|In\rangle$ $-\boxed{Z}-$ $|Out\rangle$
- Pauli matrices are involutory[1] (i.e. square of Pauli matrix is identity matrix): $I^2 = X^2 = Y^2 = Z^2 = -iXYZ = I$
- Pauli matrices are anti-commute[2], e.g., $ZX = iY = -XZ$
- Pauli$_x$-gate (also called bit-flip)
  - quantum equivalent of the NOT gate for classical computers
- Pauli$_y$-gate
  - maps $|0\rangle$ to $i|1\rangle$ and $|1\rangle$ to $-i|0\rangle$
- Pauli$_z$-gate (also called phase-flip)
  - leaves the basis state $|0\rangle$ unchanged and maps $|1\rangle$ to $-|1\rangle$

deutsch: [1] selbst-invers [2] anti-kommutativ

Controlled NOT (CNOT/CX)-Gate

- "*If the control bit is set, then it flips the target bit.*"

Quantum Circuit	Table of in- & outputs	Rotation Matrix R

Quantum Circuit

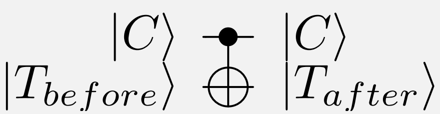


Table of in- & outputs

Inputs		Output
Control C	**Target T_{before}**	**Target T_{after}**
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

$$R \cdot \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

Rotation Matrix R

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

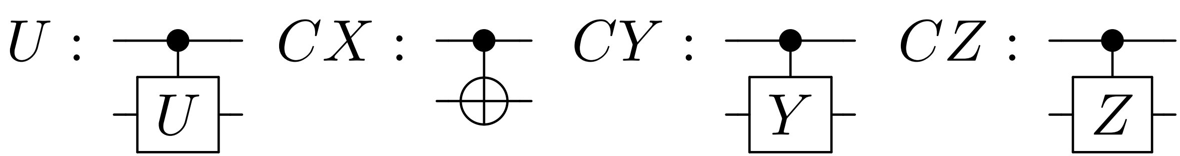
- reversible gate: 2 applications of CNOT retrieves the original input
- Classical analog of the CNOT gate is a reversible XOR gate (i.e., they have analogous in- & and outputs for $\{|0\rangle, |1\rangle\}$ inputs)
- $|a, b\rangle \mapsto |a, a \oplus b\rangle$, where $\oplus$ is XOR

Controlled-U (CU)-Gate

- "If the control bit is set, then the U transformation is applied."

$$U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}, CU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{bmatrix}$$

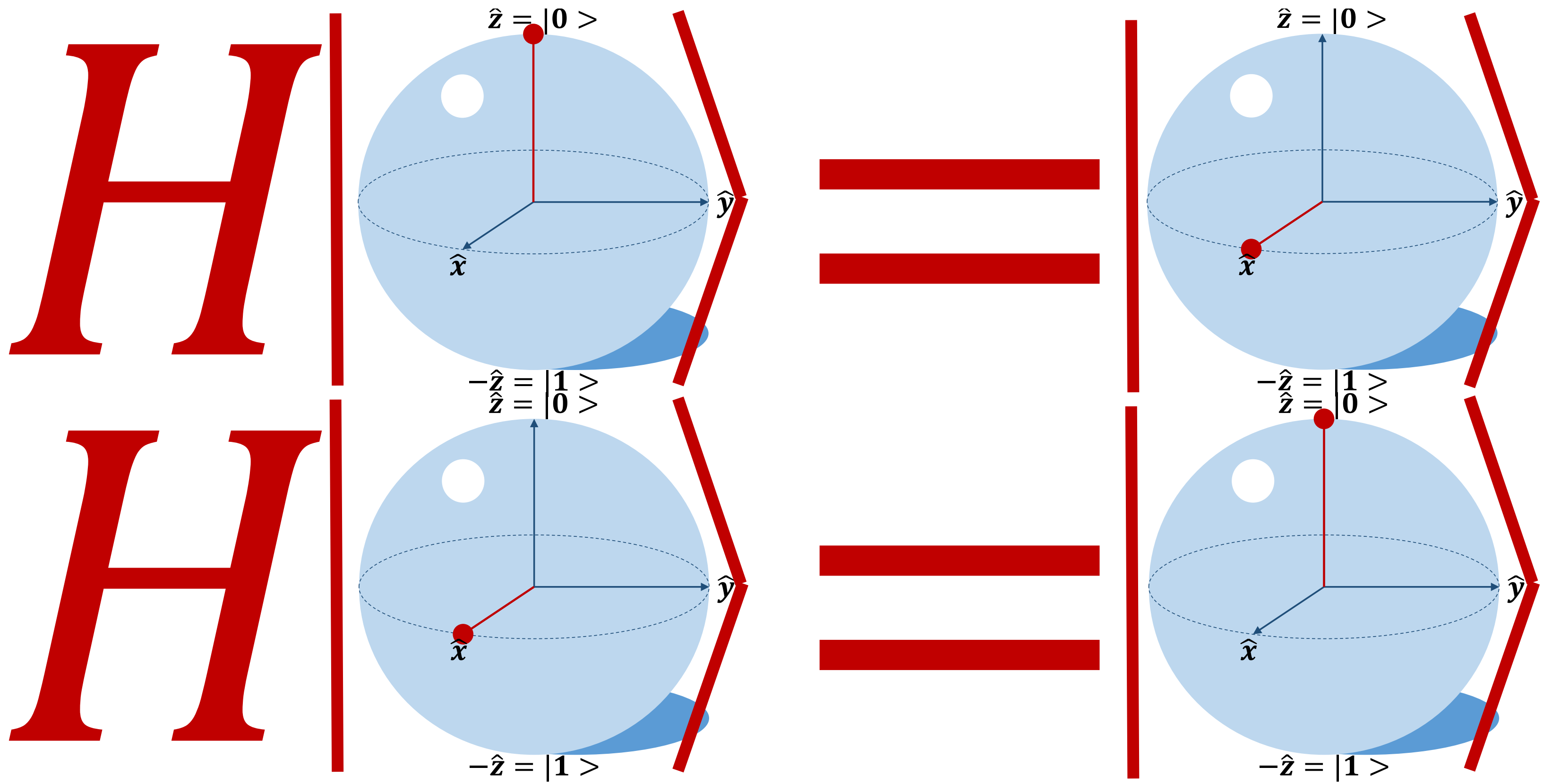
- If $U = X$: "controlled-X" (CX),
  if $U = Y$: "controlled-Y" (CY),
  if $U = Z$: "controlled-Z" (CZ)
- Circuit representations:



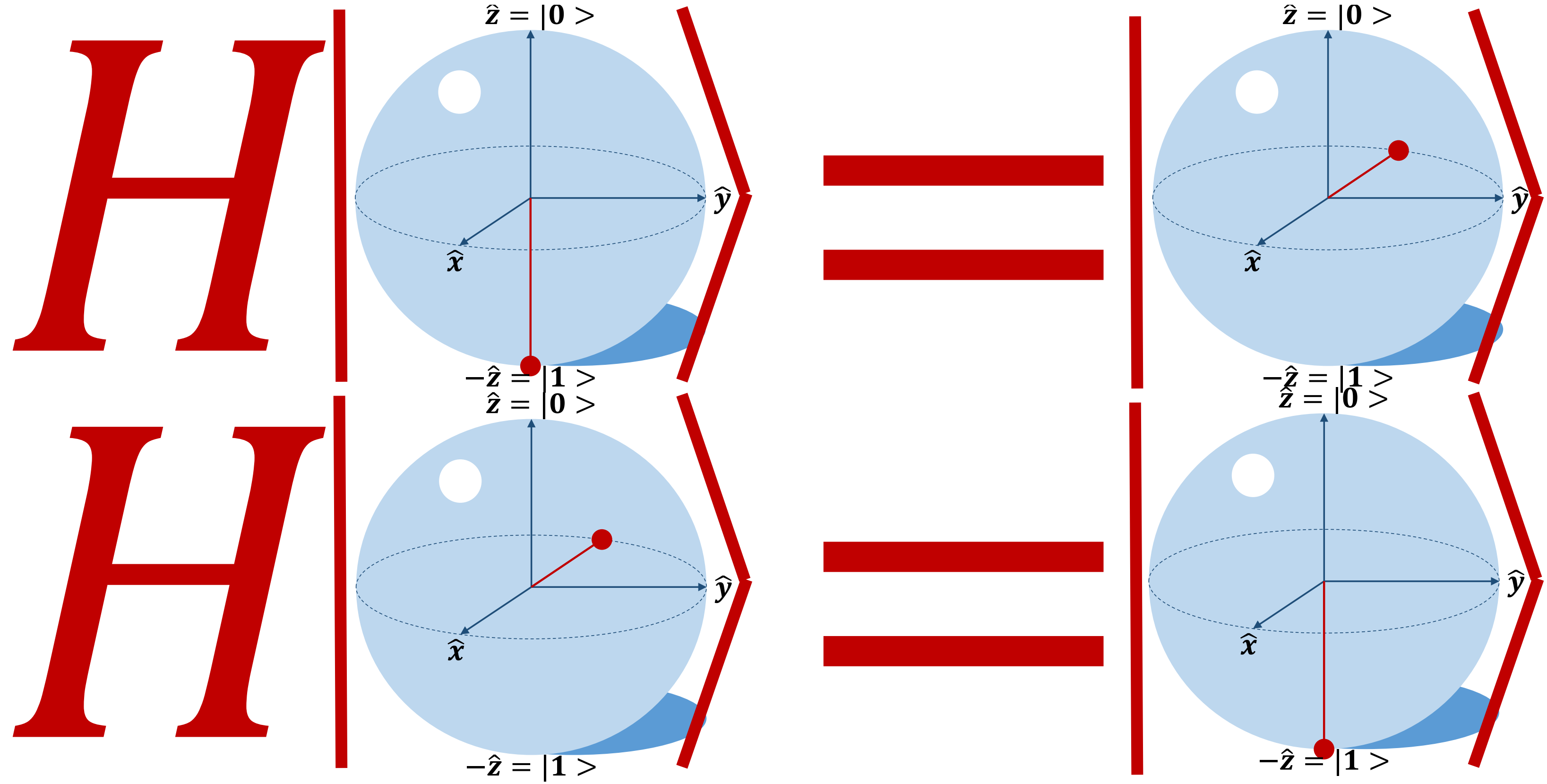
Hadamard (H)-Gate

- unitary transformation that maps qubit operations in z-axis to the x-axis and vice versa
  - represents a rotation of π about the axis $(\hat{x} + \hat{z})/\sqrt{2}$ at the Bloch sphere
  - For example, $HZH = X$, $H\sqrt{X}H = \sqrt{Z} = S$, and $HR_z(\theta)H = R_x(\theta)$
- maps the basis states $|0\rangle \mapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|1\rangle \mapsto \frac{|0\rangle-|1\rangle}{\sqrt{2}}$
  - i.e., it creates a superposition if given a basis state
- $$\boldsymbol{H} := \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix}$$
- $H^2 = I$, $R_y(\pi/2)Z = H$ and $XR_y(\pi/2) = H$
- Controlled H gate: CU-Gate with $U = H$

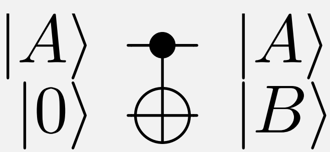
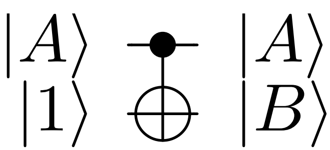
Hadamard transformation on the Bloch Sphere for $|0\rangle$



Hadamard transformation on the Bloch Sphere for $|1\rangle$



Bell States via Entanglement 1/4

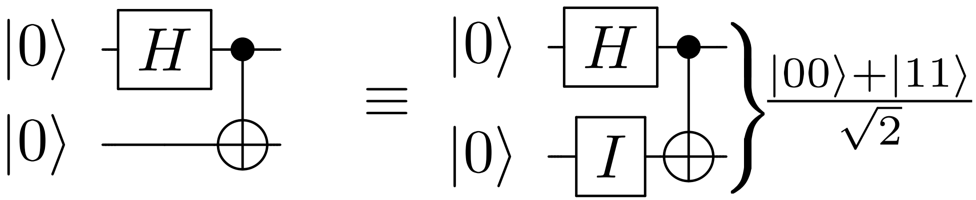
Entanglement	Quantum Circuit	Table of in- & outputs
Correlated		A / B: $\vert 0\rangle$ / $\vert 0\rangle$; $\vert 1\rangle$ / $\vert 1\rangle$; $\frac{1}{\sqrt{2}}(\vert 00\rangle + \vert 11\rangle)$
Anti-Correlated		A / B: $\vert 0\rangle$ / $\vert 1\rangle$; $\vert 1\rangle$ / $\vert 0\rangle$; $\frac{1}{\sqrt{2}}(\vert 01\rangle + \vert 10\rangle)$

- Even if the entangled qubits are at different locations, they are still entangled (phenomena appears to happen instantaneously ignoring speed of light: open question in physics)
  - As of 2017 experimentally verified for distances of up to 1200 kilometers
- Succeeding operations on entangled qubits do not change the state of the other entangled qubits if the operation is not a measurement

Yin et al., "Satellite-based entanglement distribution over 1200 kilometers". Quantum Optics. 356: 1140–1144, 2017

Bell States via Entanglement 2/4

Circuit Representation:



$$H \otimes I = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\ \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & -1\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}$$

$$CNOT(H \otimes I)|00\rangle = \left(\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}\right)\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 0\\ 0\\ 1\end{bmatrix} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Bell States via Entanglement 3/4

- How many atoms can be entangled?
  - Experiment [C+23] with 2 spatially separated Bose-Einstein condensates, each containing about 700 rubidium atoms
    - Entanglement between the condensates results in strong correlations of their collective spins
- How much time for 2 electrons being entangled? [J+24] [V+24]
  - Experiment:
    Atoms hit by a laser pulse $\Rightarrow$ Sometimes: 2 electrons will be entangled:
    1 flying away and 1 shifting into a state with higher energy
  - "The electron [...] is a wave that spills out of the atom [...] and that takes a certain amount of time," says Březinová. "It is precisely during this phase that the entanglement occurs [...]"
  - If the remaining electron is in a state of higher energy, then the electron that flew away was more likely to have been torn out at an early point in time;
    if the remaining electron is in a state of lower energy, then the 'birth time' of the free electron that flew away was likely later
    on average ≈ 232 attoseconds $= 232 \cdot 10^{-18}$ seconds

Bell States via Entanglement 4/4

- Verifying entanglement of quantum states:
  None of the Bell states can be broken into (tensor) products!
- e.g., for 2 qubits: $\begin{bmatrix} x \\ y \end{bmatrix} \otimes \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} x \cdot w \\ x \cdot z \\ y \cdot w \\ y \cdot z \end{bmatrix}$
  - $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow y = z = 0, x \cdot w = 1$, i.e., **no entanglement**!
  - $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow (x = 0 \vee w = 0) \wedge x \neq 0 \wedge w \neq 0$ ↯, i.e., **entanglement**!

Phase shift gates

- map the basis states $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto e^{i\varphi}|1\rangle$
- probability of measuring a $|0\rangle$ or $|1\rangle$ is unchanged
- modify the phase of the quantum state. This is equivalent to tracing a horizontal circle (a line of latitude) on the Bloch sphere* by φ radians
- $P(\varphi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}$, where φ is the phase shift with the period 2π
- $P(\pi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} =: Z$

  $P(\frac{\pi}{2}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \sqrt{Z} =: S$ (although sometimes used for SWAP)

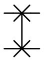
  $P(\frac{\pi}{4}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} = \sqrt[4]{Z} = \sqrt[2]{S} =: T$
- Controlled phase shift gates: CU-Gate with $U = P(\varphi)$

* deutsch: Dies entspricht dem Zeichnen eines horizontalen Kreises (einer Breitenlinie) auf der Blochkugel

Swap-Gate ($SWAP$)

- swaps two qubits:

$$SWAP := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Circuit Representation: 

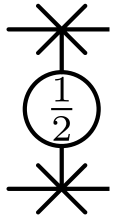
$$SWAP(\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle) = SWAP \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \\ \gamma \\ \beta \\ \delta \end{bmatrix} = \alpha|00\rangle + \gamma|01\rangle + \beta|10\rangle + \delta|11\rangle$$

Square root of Swap-Gate ($\sqrt{SWAP}$)

- arises naturally in systems that exploit exchange interaction*
- universal gate such that any many-qubit gate can be constructed from only $\sqrt{SWAP}$ and single qubit gates
- not maximally entangling (i.e., more than one application of it is required to produce a Bell state)
- performs half-way of a two-qubit swap:

$$\sqrt{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1+i) & \frac{1}{2}(1-i) & 0 \\ 0 & \frac{1}{2}(1-i) & \frac{1}{2}(1+i) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Circuit Representation:



Toffoli-Gate (controlled-controlled-not (CCNOT)/Deutsch gate $D(\frac{\pi}{2})$)

- "*If the first two (qu)bits are set, then the Toffoli gate flips the third (qu)bit.*", i.e., it maps (C_1, C_2, T) to $(C_1, C_2, T\ XOR\ (C_1 \wedge C_2))$
- reversible gate
- universal: for classical computation and for quantum computation when combined with the single qubit Hadamard gate

Quantum Circuit

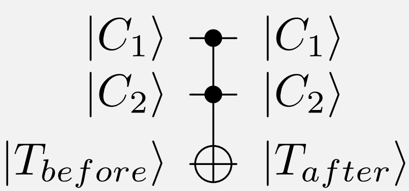


Table of in- & outputs

C_1	C_2	T_{before}	T_{after}
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$
$\frac{1}{\sqrt{2}} \cdot (\|0\rangle + \|1\rangle)$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$

Rotation Matrix R

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Fredkin-Gate ($CSWAP/CS$)

- 3-bit gate that performs a controlled swap
- universal for classical computation

Quantum Circuit

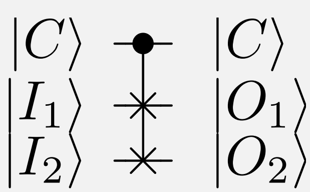


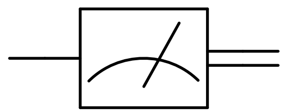
Table of in- & outputs

C	I_1	I_2	O_1	O_2
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\frac{1}{\sqrt{2}} \cdot (\|0\rangle + \|1\rangle)$	$\frac{1}{\sqrt{2}} \cdot (\|0\rangle + \|1\rangle)$	$\|0\rangle$

Rotation Matrix R

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

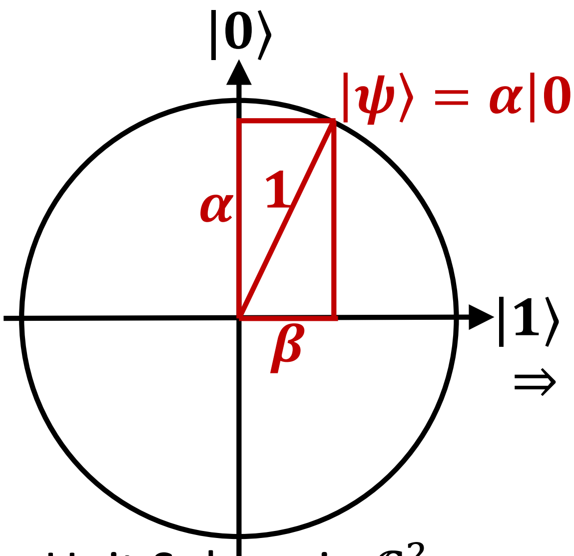
Measurement (also called observation)

Input		Circuit Representation	Output	
Information Unit	Line		Line	Information Unit
Qubit	Single		Double	Classical Bit

- irreversible and therefore not a quantum gate, because it assigns the observed quantum state to a single value

Measurement - Probabilities for basis state collapses

1 Qubit



Unit Sphere in $\mathbb{C}^2$

$$|\boldsymbol{\psi}\rangle = \boldsymbol{\alpha}|\mathbf{0}\rangle + \boldsymbol{\beta}|\mathbf{1}\rangle$$

$$\Rightarrow |\alpha|^2 + |\beta|^2 = 1$$

Measuring:

$$P(|0\rangle) = |\alpha|^2$$
$$P(|1\rangle) = |\beta|^2$$

n Qubits: Generalization to $\mathbb{C}^{2^n}$

$$|\boldsymbol{\psi}\rangle = \sum_{x=0}^{2^n-1} \boldsymbol{\alpha}_x\, |\boldsymbol{x}\rangle \in \mathbb{C}^{2^n}$$

$$\Rightarrow \sum_{x=0}^{2^n-1} |\alpha_x|^2 = 1$$

Measuring:

$$P(|x\rangle) = |\alpha_x|^2$$

Set of Universal Quantum Gates

- is any set of gates to which any operation can be reduced
  - In other words: any other unitary operation can be expressed as a finite sequence of gates from this set
- Examples of universal quantum gates sets
  - Rotation operators $R_x(\theta), R_y(\theta), R_z(\theta)$, the phase shift gate $P(\varphi), CNOT$
  - $CNOT, H, S$ and T gates
  - Two-gate set of universal quantum gates: Toffoli and Hadamard gates
  - ...

Unitary Inversion † of Gates 1/2

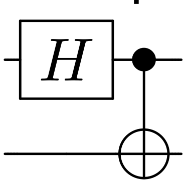
- All quantum logical gates are reversible $\Rightarrow$ any composition of multiple gates is also reversible
- Series and parallel combinations of unitary matrices are also unitary matrices $\Rightarrow$ Inversing all algorithms and functions containing only gates is possible
  - Inversion not possible for initialization, measurement, I/O and spontaneous decoherence
- U is a unitary matrix $\Rightarrow U^\dagger U = UU^\dagger = I$ and $U^\dagger = U^{-1}$
  - The dagger $\dagger$ denotes the conjugate transpose (also called the Hermitian adjoint, deutsch: adjungierte/hermitesch transponierte/transponiert-konjugierte Matrix)
- used in uncomputation* for cleaning up temporary effects on ancilla (i.e., auxiliary) bits so that they can be re-used

* S. Aaronson, D. Grier, L. Schaeffer. The Classification of Reversible Bit Operations. arXiv:1504.05155, 2015

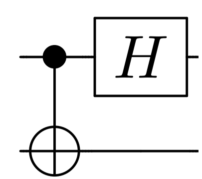
Unitary Inversion † of Gates 2/2

- Hermitian (also called self-adjoint operators[1]): Gates that are their own unitary inverses like Hadamard (H) and the Pauli gates (I, X, Y, Z)
- Skew-Hermitian[2] (also called adjoint operators): Gates that are not their own unitary inverses in general like the phase shift ($S, T, P, CPHASE$)
- $(UV)^\dagger = V^\dagger U^\dagger, (A_1 \cdots A_m)^\dagger = A_m^\dagger \cdots A_1^\dagger$
- $(U \otimes V)^\dagger = U^\dagger \otimes V^\dagger, (A_1 \otimes \cdots \otimes A_m)^\dagger = A_1^\dagger \otimes \cdots \otimes A_m^\dagger$

Example:



$^\dagger \equiv (CNOT(H \otimes I))^\dagger = (H^\dagger \otimes I^\dagger)CNOT^\dagger = (H \otimes I)CNOT \equiv$



deutsch: [1] selbst-adjungierter Operator [2] schiefhermetische Matrix

Circuit composition - Serially Wired Gates

- When gate B is put after gate A in a series circuit, then the effect of the two gates can be described as a single gate $C = B \cdot A$
- Example:

$$|\psi_1\rangle \; -\boxed{Y}-\boxed{X}- \; |\psi_2\rangle$$

$$\equiv$$

$$|\psi_1\rangle \; -\boxed{C = X \cdot Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = iZ}- \; |\psi_2\rangle = XY\,|\psi_1\rangle = C\,|\psi_1\rangle$$

Exponents of Quantum Gates

- All quantum gates are unitary matrices
- Positive integer exponents are equivalent to sequences of serially wired gates
  - e.g., $X^3 = X \cdot X \cdot X$
- Real exponents is a generalization of the series circuit
  - All real exponents of unitary matrices are also unitary matrices/quantum gates
  - e.g., X^π and $\sqrt{X} = X^{\frac{1}{2}}$ are valid quantum gates
- $U^0 = I$ for any unitary matrix U
- $U^{-n} = (U^n)^\dagger$
  - e.g., $T^{-1} = T^\dagger$ and $T^{-2} = (T^2)^\dagger = S^\dagger$

Circuit composition - Parallel Gates

- The tensor product (or Kronecker product) of two quantum gates is the gate that is equal to the two gates in parallel

  Example:

$$
\begin{array}{lcl}
|\psi\rangle & -\boxed{Y}- & Y|\psi\rangle \\
|\phi\rangle & -\boxed{X}- & X|\phi\rangle
\end{array}
\equiv
\left.\begin{array}{l}
|\psi\rangle \\
|\phi\rangle
\end{array}
-\boxed{Y \otimes X}-
\right\}
(Y \otimes X)\,|\psi \otimes \phi\rangle
$$

$$
C = Y \otimes X = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & -i \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ i \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}
$$

Circuit composition - Parallel Hadamard

- Parallel application of Hadamard gates on n qubits (all in basis state $|0\rangle$):

$$\underbrace{H \otimes H \otimes \cdots \otimes H}_{n \text{ times}} = \bigotimes_1^n H = H^{\otimes n} = H_n$$

$$\bigotimes_1^n (H|0\rangle) = \left(\bigotimes_1^n H\right)\left(\bigotimes_1^n |0\rangle\right) = \frac{1}{\sqrt{2^n}}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = \frac{|0\rangle+|1\rangle+\cdots+|2^n-1\rangle}{\sqrt{2^n}} = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle$$

- This state is a uniform superposition*:
  - If not measured, it is a quantum state with equal probability amplitude $\frac{1}{\sqrt{2^n}}$ for each of its possible states
  - Measuring this state results in a random number between $|0\rangle$ and $|2^n - 1\rangle$
  - Often first step of quantum computing algorithms like Grover's search

* deutsch: gleichmäßige Überlagerung

Computational Complexity of Simulating Quantum Computers

- Time complexity for multiplying two $k \times k$-matrices is at least $\Omega(k^2 \log k)$ on a classical machine*
- For n-qubits-gate: $k = 2^n$
  $\Rightarrow$ intractable to simulate large entangled quantum systems using classical computers
- Efficient simulation of subsets of the gates, such as the Clifford gates, or combinations of $X, CNOT$, Toffoli
- State vector of a quantum register is 2^n complex entries
  $\Rightarrow$ Storing the probability amplitudes as a list of floating point values is not tractable for large n

* R. Raz, On the complexity of matrix product. STOC, 2002

More Quantum Logic Gates...

- Ising coupling gates
  - implemented natively in some trapped-ion quantum computers
    - Debnath et al. Demonstration of a small programmable quantum computer with atomic qubits. Nature 536, 63–66,2016
- Imaginary swap ($iSWAP$) and it root version $\sqrt{iSWAP}$
  - for systems with Ising like interactions
    - S.E. Rasmussen, N.T. Zinner. Simple implementation of high fidelity controlled- i swap gates and quantum circuit exponentiation of non-Hermitian gates. Physical Review Research. 2(3), 2020
    - N. Schuch, J. Siewert. Natural two-qubit gate for quantum computation using the XY interaction. Physical Review A. 67(3), 2003
    - P.-L. Dallaire-Demers, F.K. Wilhelm. Quantum gates and architecture for the quantum simulation of the Fermi-Hubbard model. Physical Review A. 94 (6), 2016
- Deutsch gate (named after physicist David Deutsch)
  - some proposals to realize a Deutsch gate with dipole-dipole interaction in neutral atoms
    - X.-F. Shi. Deutsch, Toffoli, and cnot Gates via Rydberg Blockade of Neutral Atoms. Physical Review Applied. 9(5), 2018
- ...

COVID-19 Variant of the El Farol Bar Problem for simplicity reasons for 4 friends

- 4 friends want to go to a restaurant for lunch.
- However, because of COVID-19 confinements, only 3 guests are allowed to sit at a table of a restaurant.
- If all friends go to this restaurant, then no one will go inside because of solidarity, because one of them would stay hungry if 3 of them go into the restaurant.
- If 3 go to restaurant, then these 3 have most fun compared to only 1, 2 or none of them go to restaurant.
- The friends do not want to decide which of them goes to the restaurant. Therefore, they independently decide at home without communication whether or not to go to the restaurant.

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After some trials the friends are not satisfied with the procedure. One of the friends studied quantum information theory and proposes: **"Let's decide it by a quantum computer!"**

COVID-19 Variant of the El Farol Bar Problem for simplicity reasons for 4 friends

- k-th friend gets the k-th qubit. If the k-th qubit is set, then the k-th friend stays at home and the other friends go to restaurant
- Hence the quantum circuit should deliver one of $|0001\rangle, |0010\rangle, |0100\rangle, |1000\rangle$ with equal probability
  $\rightsquigarrow$ (General) W-State (for n qubits)

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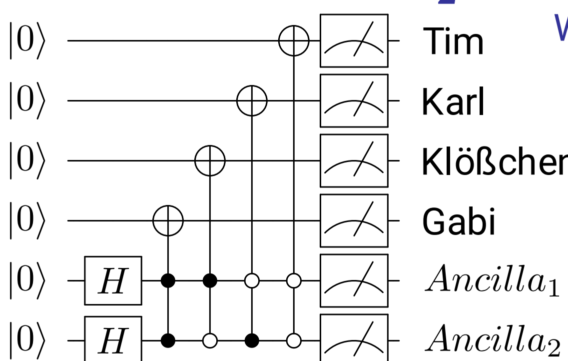
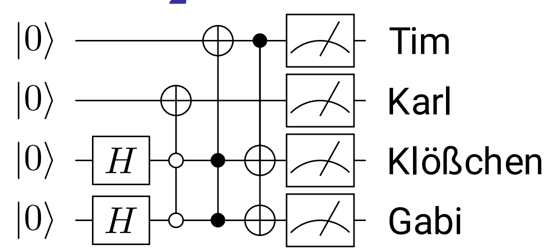
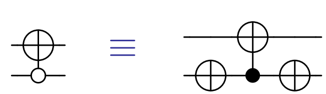
With ancilla:



Without ancilla:



Legend:



Summary and Conclusions

- Quantum Logic Gates
  - Rotation: Pauli (I, X, Y, Z), $R_x, R_y, R_z, H, P(\varphi)$
  - $SWAP, \sqrt{SWAP}$
  - Controlled Gates: $CX, CU, CCNOT, CSWAP$
  - Measurement
  - ...
- Bell States via Entanglement
- Universal Quantum Logic Gates
- Unitary Inversion
- Circuit
  - Composition
  - Simulation