Lecture

Quantum Computing

(CS5070)

Quantum Data Encoding Patterns

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Motivation Design Patterns

- Software design patterns
  - structured document containing an abstract description of a proven solution of a recurring problem
  - refers to other patterns that may jointly contribute to an encompassing solution of a complex problem

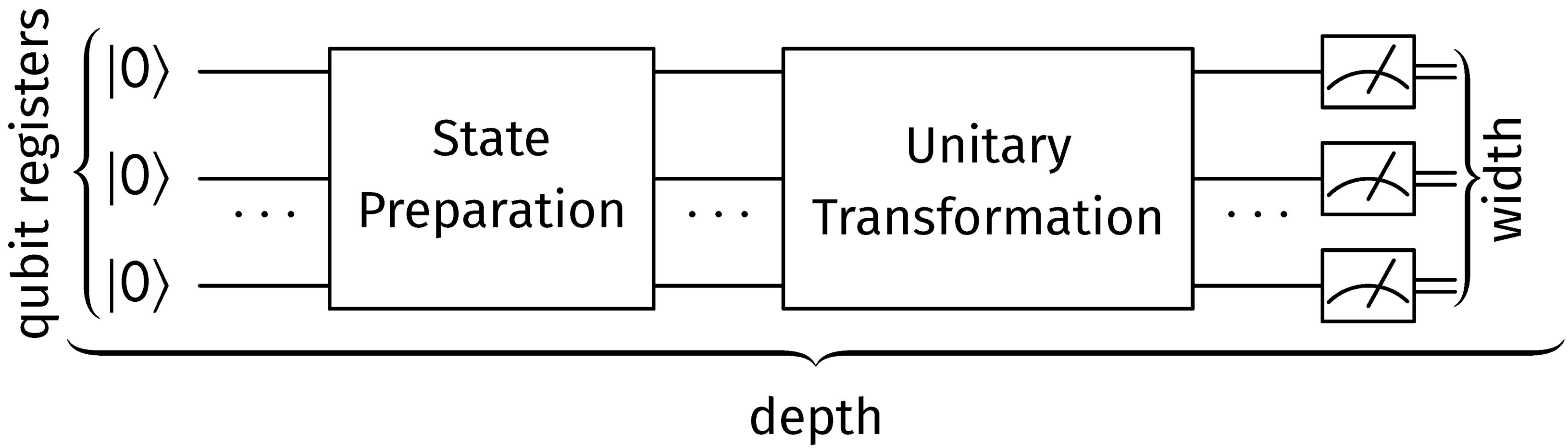
    ⇒ network of related patterns, i.e. a pattern language
  - **Example:** Singleton, Definition: **Link 1** **Link 2**
    - creational design pattern that lets you ensure that a class has only one instance, while providing a global access point to this instance
    - **Solution**
      - Make the default constructor private, to prevent other objects from using the new operator with the Singleton class
      - Create a static creation method that acts as a constructor. Under the hood, this method calls the private constructor to create an object and saves it in a static field. All following calls to this method return the cached object.
- Quantum Patterns
  - Design patterns using quantum algorithms

Pattern Format

- differs depending on the domain
- Format example:
  - Name and icon that serves as a graphical representation of the pattern
  - Intent that briefly summarizes the purpose of the pattern
  - Alias: other used names of the pattern
  - Context: problem and the circumstances of the pattern
  - Forces: trade-offs or considerations that must be taken into account for solving the problem
  - Solution: description in an abstract manner and often visualized by a solution sketch
  - Result: consequences of the solution
  - optional section for variants of the pattern
  - Related patterns: Connections between patterns, as patterns are often applied in combination or solve similar problems
  - Known uses of the pattern in quantum algorithms and concrete implementations

Typical Structure of a Quantum Calculation

- Typical phases of quantum calculations:



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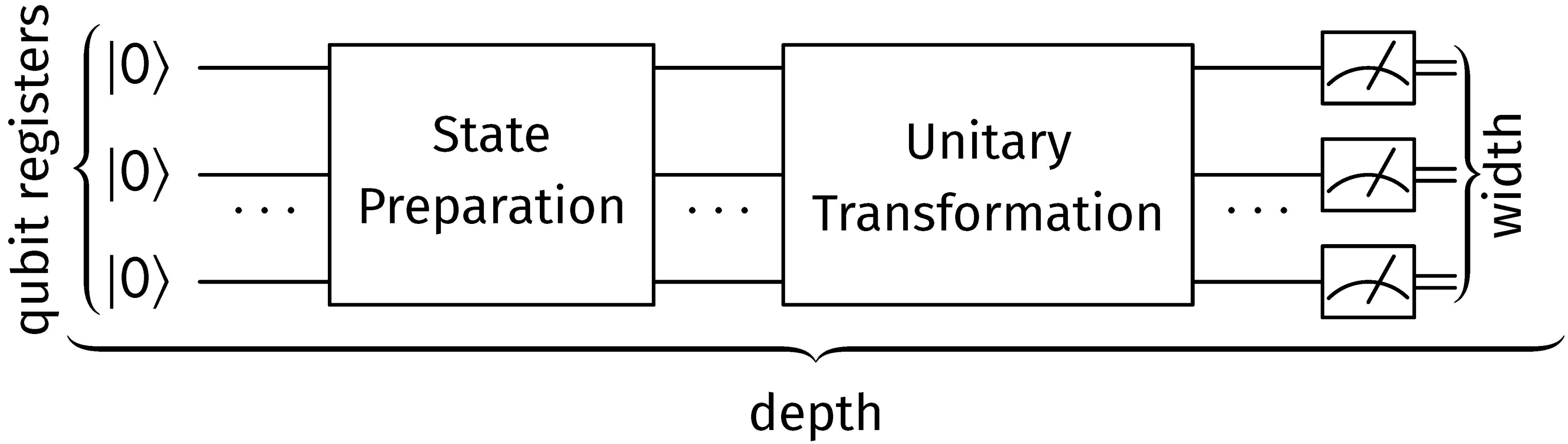
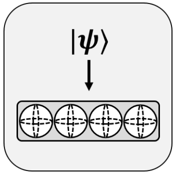


- **How to prepare the states for data loading?**



Initialization aka State Preparation 1/2

- Intent
  - Initialize the input of a quantum register, taking into account the prerequisites of the subsequent steps of the algorithm
- Context
  - Specific parameters must be typically given as input data to a quantum algorithm in order to solve a given problem
  - The first unitary transformations of a quantum circuit typically encode the input data into the quantum register according to a defined encoding
- Solution
  - $|0\cdots0\rangle$ and $|0\cdots01\rangle$ are frequently used as initialization of quantum registers
  - Some qubits as ancilla bits, which may be used for the storage of intermediate results or quantum error correction
  - Example
    - for a function table of a Boolean function $f:\{0,1\}^n \rightarrow \{0,1\}^m$, the overall register is initialized as $|0\rangle^{\otimes n}\,|0\rangle^{\otimes m}$ (including m ancilla bits)

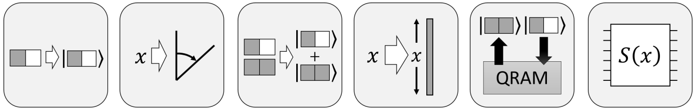
Initialization aka State Preparation 2/2

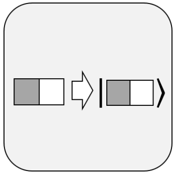
- Result
  - Preparing more advanced states built on the previously described initialization techniques
  - Examples of loading into quantum registers
    - Loading classical bits[1]
    - Loading of complex vectors[2]
    - Loading of real-valued vectors[3]
    - Loading of matrices that are represented as sets of vectors[4]
- Related Patterns
  - Uniform superposition: $H_n(|0\cdots0\rangle) = \frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1}|i\rangle$
  - Refinements of initialization:
  - Initialized register may be used to compute a function table

For other patterns we do not present all items...

Data Encoding

- define how data is represented by the state of a quantum system
- Data encoding patterns describe a particular encoding as a trade-off between:
  - Minimize #qubits needed for the encoding
    - because current devices are of intermediate size and thus only support a limited #qubits
  - Minimize depth of quantum circuit needed to realize the encoding
    - the loading routine is ideally of constant or logarithmic complexity
  - Data must be represented in a suitable manner for further calculations, e.g., arithmetic operations
- Patterns for Data Encoding:



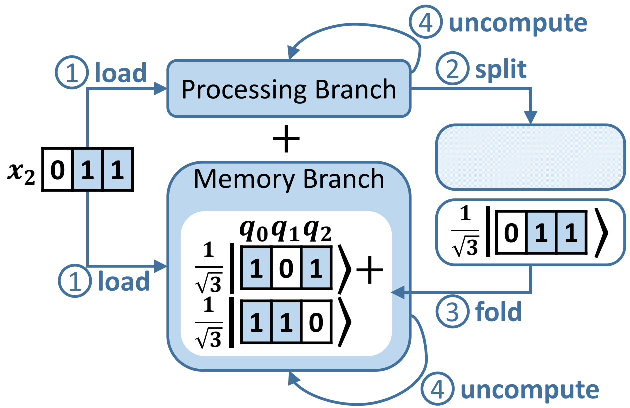


Basis Encoding

- Intent
  - Represent data in a quantum computer for performing calculations
- Context
  - Quantum algorithm requires numerical input data X for further calculations
- Solution
  - use the computational basis $|0...00\rangle, |0...01\rangle, \ldots, |1...11\rangle$
    - For example: decimal 2 in binary format 10 encoded by $|10\rangle$
    - Classical data: integer number $x := b_{n-1} \ldots b_1 b_0$ with n bits b_i
    - Corresponding quantum data: $|x\rangle := |b_{n-1} \ldots b_1 b_0\rangle$
- Result
  - suitable for arithmetic computations [LB'20][VBE'96][CB'18] $\Rightarrow$ digital encoding
  - Space requirements: n qubits
  - Initial $|0\rangle$ state of qubits that represent a 1 bit must be flipped into $|1\rangle$ $\Rightarrow O(n)$ NOT-gates in parallel $\Rightarrow O(1)$ preparation time

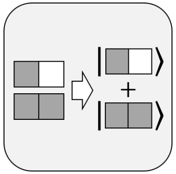
Quantum Associative Memory (QuAM)

- Context
  - A quantum algorithm requires multiple numerical values $X := \{x_1, \ldots, x_k\}$ as input for further calculations.
- Solution
  - Use a quantum associative memory (QuAM) [VM'00] to prepare a superposition of basis encoded values in the same qubit register [LB'20] $\Rightarrow$ quantum register is an equally weighted superposition $\frac{1}{\sqrt{k}} \sum_{i=1}^{k} |x_i\rangle$ of all basis encoded values $|x_1\rangle, \ldots, |x_k\rangle$



Basic steps [VM'00] (using modified parts of Grover):

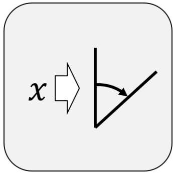
(1) Additional element is prepared in both branches.
(2) Processing branch is split in such a manner, that the new element gets a proper amplitude such that
(3) it can be brought into superposition with the already added elements.
(4) Uncompute cleans for the next iteration.



Quantum Associative Memory (QuAM)

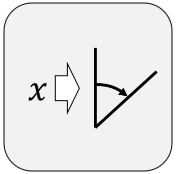
- Result
  - digital encoding and therefore suitable for arithmetic computations [LB'20]
  - Space: For k input numbers each with n bits, n qubits are needed
  - Each of encoded input values is represented by a basis vector with an amplitude of $\frac{1}{\sqrt{k}}$ and all other amplitudes of the register are 0

    $\Rightarrow$ The amplitude vector therefore often sparse [SP'18]
  - Preparation time depends on number k of input numbers



Angle Encoding 1/3

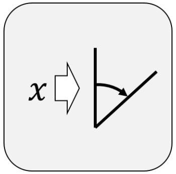
- Intent
  - "Represent each data point by a separate qubit" [WBLS'21]
- Alias
  - Qubit Encoding: since each qubit represents a single data point [LC'20]
  - (Tensor) Product Encoding: since the resulting encoding of this pattern is not entangled [LB'20]
- Context
  - Encoding is efficient in terms of operations to perform more operations within the decoherence time of noisy intermediate-scale quantum computers (NISQ) after encoding the data



Angle Encoding 2/3

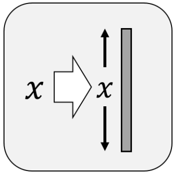
- Solution [LB'20]
  1. Each data point of the input is normalized to the interval $[0, 2 \cdot \pi]$ (which ensures an injective encoding, i.e., $\forall a, b : f(a) = f(b) \Rightarrow a = b$, because rotation gates of the form $R_{\{x,y,z\}}(2 \cdot x_i)$ are periodic with a period of $2 \cdot \pi$)
  2. Encoding the data points by rotating around the y-axis with an angle depending on the value of the normalized data point

$$
\left.
\begin{array}{l}
|0\rangle - \boxed{R_y(2 \cdot x_0)} - \begin{bmatrix} cos(x_0) \\ sin(x_0) \end{bmatrix} \\
|0\rangle - \boxed{R_y(2 \cdot x_1)} - \begin{bmatrix} cos(x_1) \\ sin(x_1) \end{bmatrix} \\
\vdots \qquad \vdots \\
|0\rangle - \boxed{R_y(2 \cdot x_n)} - \begin{bmatrix} cos(x_n) \\ sin(x_n) \end{bmatrix}
\end{array}
\right\} \equiv \begin{bmatrix} cos(x_0) \\ sin(x_0) \end{bmatrix} \otimes \begin{bmatrix} cos(x_1) \\ sin(x_1) \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} cos(x_n) \\ sin(x_n) \end{bmatrix}
$$



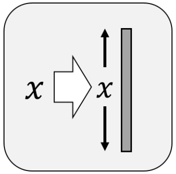
Angle Encoding 3/3

- Result
  - Space requirements: $n+1$ qubits for $n+1$ data points
  - Initial $|0\rangle$ state of qubits that represent a 1 data point must be rotated according to data point
    $\Rightarrow O(n)$ Rotation-gates in parallel $\Rightarrow O(1)$ preparation time
- Variants
  - [LC'20] proposes to exploit the relative phase for a more dense encoding which requires only half of the qubits for the same amount of data points
- Known Uses
  - Classification algorithms [LC'20,G+'18] based on angle encoding
  - Quantum image processing **[YIV'15]**: angle encoding for a pixel's color information in the flexible representation of quantum image (FRQI) and an additional register for the position
  - In quantum neural networks **[SSP14]** quantum neurons (quron) use angle encoding



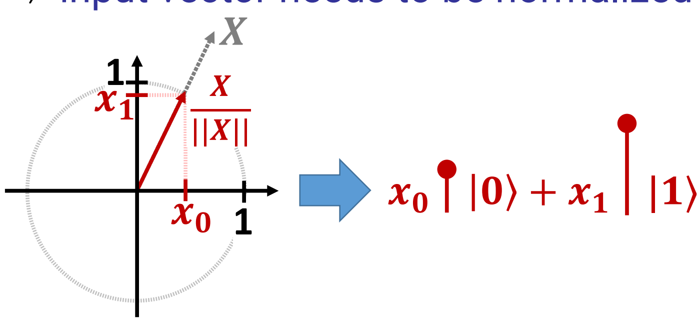
Amplitude Encoding

- Intent
  - Encode data in a compact manner that do not require calculations
- Alias
  - Wavefunction Encoding [LC'20]
    - Every quantum system is described by its wavefunction ψ defining also the measurement probabilities
      $\Rightarrow$ amplitudes of the quantum system represent data values
- Context
  - Encoding of a numerical input data vector $(x_0, \dots, x_n)^T$ for a quantum algorithm.

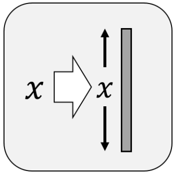


Amplitude Encoding

- Solution
  - Encoding of the input vector in the amplitudes of the quantum state: $|\psi\rangle = \sum_{i=0}^{n} x_i |i\rangle$
    - Squared moduli of the amplitudes of a quantum state must sum up to 1
      $\Rightarrow$ input vector needs to be normalized to length 1



    - Vector space of an n qubit register has dimension 2^n
      $\Rightarrow$ input vector can be padded with additional zeros if dimension is not a power of 2
    - Amplitudes depend on the data $\Rightarrow$ process of encoding the data (but not the encoding itself) is often referred to as *arbitrary state preparation*

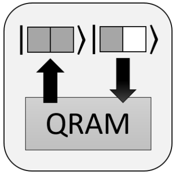


Amplitude Encoding

- Result
  - Compact representation: $\lceil log_2(n+1) \rceil$ qubits
    - more compact (in terms of qubits) than Basis, Angle Encoding or Quantum Random Access Memory (QRAM) Encoding
  - For an arbitrary state represented by k qubits (i.e., 2^k data values), at least 2^k parallel operations for initialization are needed [SP'18] (nearly reached in state-of-the-art approaches)
  - For special cases logarithmic runtime or $O(1)$ (e.g., Uniform Superposition)
  - Sparse data vectors can also be prepared more efficiently [SP'18]
  - Output is often also encoded in the amplitude
    $\Rightarrow$ Multiple measurements to obtain a good estimate of the output result
    $\Rightarrow$ Number of measurements scales with the number of amplitudes 2^k for k qubits [SP'18]

Amplitude Encoding

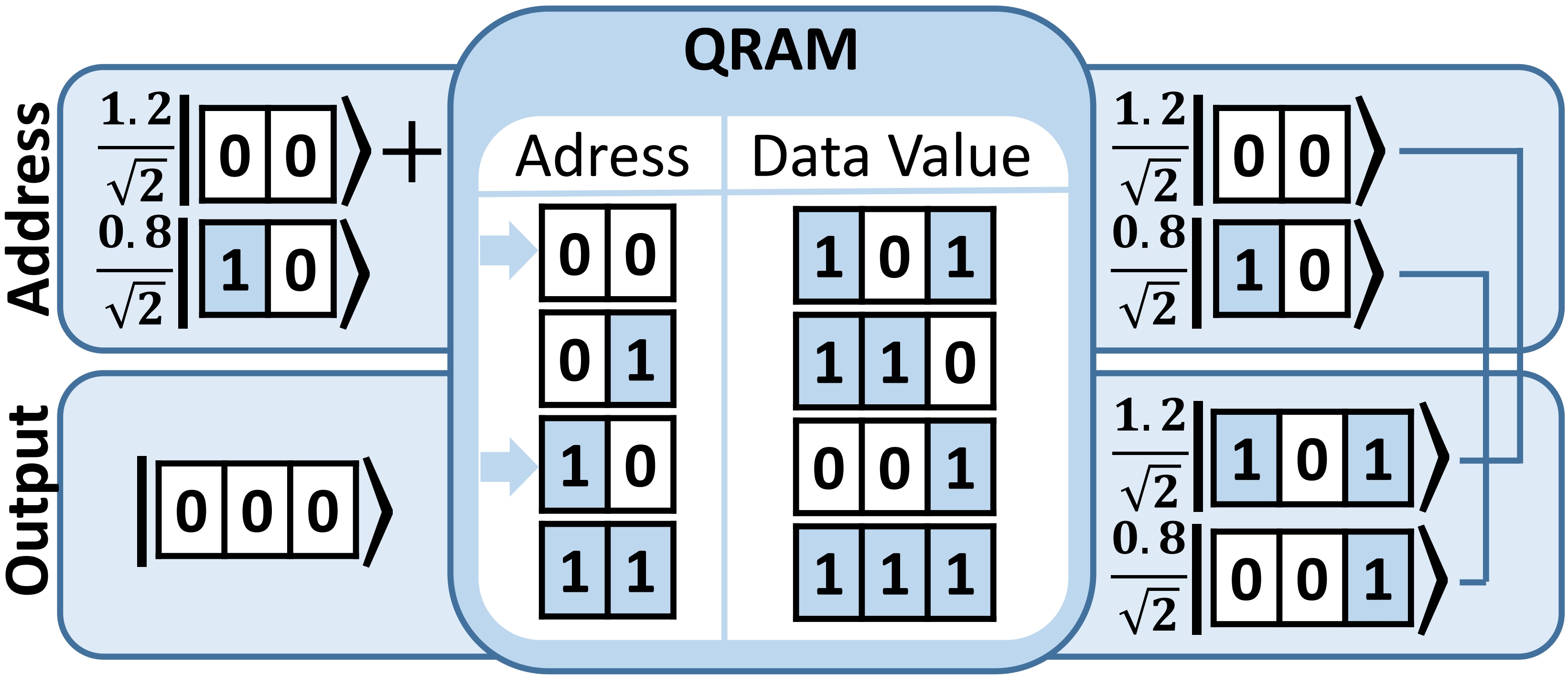
- Known Uses
  - Amplitude Encoding can be used in many quantum machine learning algorithms [LC'20]
  - Algorithm of Harrow, Hassidim and Lloyd [HHL'09] (HHL algorithm) for solving linear equations
  - Data values are typically normalized in machine learning [SFP'17], e.g. in support vector machine.
  - Various ways to construct a state preparation routine for amplitude encoding via e.g. Schmidt Decomposition **[PB'11]** ↗ **(ArXiv)** ↗ **[I+16]** ↗ **(ArXiv)** ↗
    - Mathematica: **[I+'19]** ↗
    - Qiskit: **[SBM'06]** ↗ **(ArXiv)** ↗ **QisKit Documentation** ↗
    - PennyLane: **qml.AmplitudeEmbedding** ↗ using the algorithm proposed by **[MV'05]** ↗ requiring an exponential number of operations to encode 2^k data values
    - Q#: approximates the desired amplitude encoding **Q# API reference** ↗



QRAM Encoding 1/4

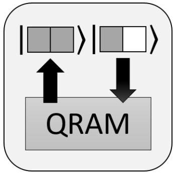
- Intent
  - "Use a quantum random access memory to access a superposition of data values at once" [W+'21]
- Context
  - Accessing the values of input data via random access memory
- Solution
  - **Classical** random access memory (RAM) transfers the data value stored at a given address into a specified output register
  - **Quantum** random access memory (QRAM): similar to RAM, but the registers are not classical but quantum registers [JHG'19]
  - Consequently, address and output registers can be in superposition instead of classical values

QRAM Encoding 2/4



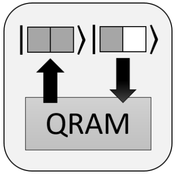
$$\sum_a c_a|a\rangle \xrightarrow{\text{QRAM}} \sum_a c_a|a\rangle|D_a\rangle$$

with c_a amplitude, $|a\rangle$ address and $|D_a\rangle$ data value of address $|a\rangle$



QRAM Encoding 3/4

- Result
  - Data values consuming n bits: n qubits for basis encoded data
  - Address register: additional $\lceil log(k) \rceil$ qubits for up to k addresses
  - Basis Encoding is used for data values: computational properties $\approx$ other digital encodings (e.g., QuAM and Basis Encoding):
    - Since data values are represented in superposition,
      - data values can be manipulated at once (using quantum parallelism)
      - multiple arithmetic operations (e.g., $+, \cdot$) can be applied
  - State preparation via the QRAM is efficient and of logarithmic runtime [SP'18]: QRAM queries N adresses in $O(log(N))$ [KKP'20]
    - Exponential speed-up of an algorithm using QRAM: only possible if filling QRAM is efficient
  - To our best knowledge, there are currently **no** commercial **hardware** implementations for QRAM
    - State preparation routine must be used for loading the QRAM, but **no** routine for arbitrary input data exists that is as efficient as QRAM

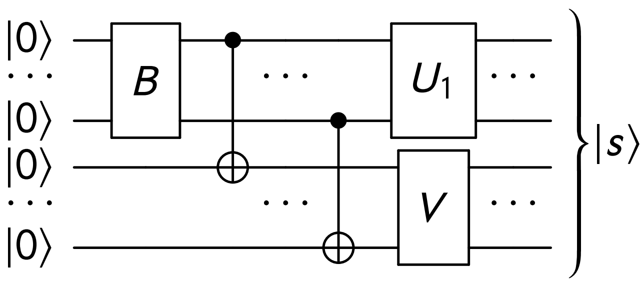


QRAM Encoding 4/4

- Known Uses
  - Alternative state preparation to realize QRAM Encoding can be found in [CB'18] (circuit family #3) or [P'14]
  - Algorithms for solving semi-definite programs [MKF'19] use QRAM Encoding.
  - QRAM is required or assumed in various other algorithms [GLM'08], **[RML'14]**, **[WKS'14]**, **[LMR'13]**
  - HHL algorithm for solving linear equations **[HHL'09]** uses QRAM Encoding as an intermediate representation for eigenvalues [MKF'19]

Schmidt Decomposition

- Context
  - A state $|s\rangle$ has to be prepared on an empty n-qubit register
- Forces
  - Small depth of the constructed state preparation circuit
  - Runtime for constructing the state preparation circuit on classical computer should not outweigh the potential benefit of quantum computing
- Solution
  - Determine B, U_1, V & apply circuit for state preparation of given $|s\rangle$ [A+20]:



Example 1:

$$|s\rangle = \frac{|00\rangle + |01\rangle}{\sqrt{2}}$$

$$B = U_1 = I$$

$$V = H$$

Example 2:

$$|s\rangle = \frac{\sqrt{2} \cdot |00\rangle + |11\rangle}{\sqrt{3}}$$

$$B = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & 0 \\ 1 & 0 \end{bmatrix}$$

$$U_1 = V = I$$

  - For the execution on a quantum computer, the unitary matrices B, U_1, V must be further decomposed into one and two qubit gates [LB'20]

Schmidt Decomposition

- Solution (continued)
  - How to determine B, U_1, V?
    - Express $|s\rangle$ in terms of two subspaces V and W that span $H^{\otimes n}$
      - Choose orthogonal basis $\{f_1, \ldots, f_k\} \in V \wedge \{g_1, \ldots, g_k\} \in W$, such that: $|s\rangle = \sum b_{ij} \cdot f_i \otimes g_j$
        - Examples: $f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, g_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
          - Example 1: $|s\rangle = \frac{|00\rangle+|01\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot f_1 \otimes g_1 + \frac{1}{\sqrt{2}} \cdot f_1 \otimes g_2$
          - Example 2: $|s\rangle = \frac{\sqrt{2}\cdot|00\rangle+|11\rangle}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot f_1 \otimes g_1 + \frac{1}{\sqrt{3}} \cdot f_2 \otimes g_2$
    - $M := \{b_{ij}\}$
      - Example 1: $M = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, Example 2: $M = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$

$S(x)$ Schmidt Decomposition

- Solution (continued)
  - How to determine B, U_1, V?
    - Compute the singular value decomposition (SVD)
      $M = \left(U_1 U_2\right)\left(\begin{matrix} A \\ 0 \end{matrix}\right) V^*$ of M (see [OS'18] for detailed instructions)
      - Example 1: Please see this link for details
        $$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^*$$
    - Entries $\{\alpha_1, \ldots, \alpha_m\}$ of the diagonal matrix A: Schmidt decomposition
      - Example 1: Schmidt decomposition is $\{1, 0\}$
    - With $\alpha_1, \ldots, \alpha_m$ Schmidt coefficients for the Schmidt basis $\{u_i\}, \{v_i\}$:
      $|s\rangle = \left(U_1 \otimes V\right) \sum_{i=1}^{m} \alpha_i \cdot e_i \otimes e_i = \sum_{i=1}^{m} \alpha_i \cdot u_i \otimes v_i, \alpha_i \in \mathbb{R} \geq 0$, where
      $\sum_{i=1}^{m} \alpha_i = 1$ (≈ The decomposition that minimally entangles the two subsystems...)
      - Example 1: $|s\rangle = 1 \cdot |0\rangle \otimes \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle) + ~~0 \cdot |1\rangle \otimes \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle)~~$
        Remark: $V = H$ can be also used because of 0

Schmidt Decomposition

- Solution (continued)
  - Example 2:
    - Please see this link how to compute the singular value decomposition for example 2:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^*$$

    - $|s\rangle = \frac{\sqrt{2}}{\sqrt{3}} \cdot |0\rangle \otimes |0\rangle + \frac{1}{\sqrt{3}} \cdot |1\rangle \otimes |1\rangle$
  - Remaining steps
    - B transforms the amplitude of the first register to the Schmidt coefficients
    - Copy this state to the second register using CNOT operations
    - U_1 and V transform the basis states $\{e_i\}$ into the Schmidt basis states

$S(x)$ Schmidt Decomposition

- Result
  - The state $|s\rangle$ is created in the register for which the Schmidt coefficients α_i are known, which can be used to quantify entanglement [NC'10]
  - The state $|s\rangle$ is separable (i.e., not entangled) if and only if exactly one of the Schmidt coefficients is non-zero
  - Depth of the circuit is $\frac{23}{48}2^n$ in the worst case [PB'11]
    - Arbitrary state preparation is of exponential complexity in general (in the worst case)
- Related Patterns
  - Schmidt Decomposition can be used as a state preparation method for Amplitude or QRAM Encoding
- Known Uses
  - Schmidt Decomposition can be used to create random states with a controlled amount of entanglement [DGK'14]
  - Mathematica implementation: [I+21]

Comparison Data Encoding Patterns

- k data points, one data point consumes n bits

Encoding Pattern	Encoding	#Qubits	Preparation	Digital Encoding
Basis	$x_i \approx \sum_i b_i \cdot 2^i$ $\mapsto \|x_i\rangle = \|b_{n-1} \dots b_1 b_0\rangle$	$k \cdot n$	$O(1)$	✓
Angle	$x_i \mapsto cos(x_i)\|0\rangle + sin(x_i)\|1\rangle$	k	$O(1)$	
QuAM	$X \mapsto \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \|x_i\rangle$	n	$O(k)$	✓
QRAM	$X \mapsto \sum_i c_i \|i\rangle \|x_i\rangle$	$n + \lceil log_2(k) \rceil$	$O(2^k)$ (Schmidt)	✓
Amplitude	$X \mapsto \sum_{i=0}^{k-1} x_i \|i\rangle$	$\lceil log_2(k) \rceil$	$O(2^k)$ (Schmidt)	

Summary & Conclusions

- Quantum Patterns
  - Software design patterns using quantum algorithms
- In this lecture
  - Patterns for Data Encoding
    - State Preparations refined by
      - Basis Encoding
      - Quantum Associative Memory (QuAM)
      - Angle Encoding
      - Amplitude Encoding
      - QRAM Encoding
    - Schmidt Decomposition to be used as basic state preparation method for Amplitude and QRAM Encoding