Non-Standard Databases and Data Mining

Counterfactuals

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Structural Causal Models

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Part IV: Counterfactuals

Literature

- J.Pearl, M. Glymour, N. P. Jewell: Causal inference in statistics – A primer, Wiley, 2016.

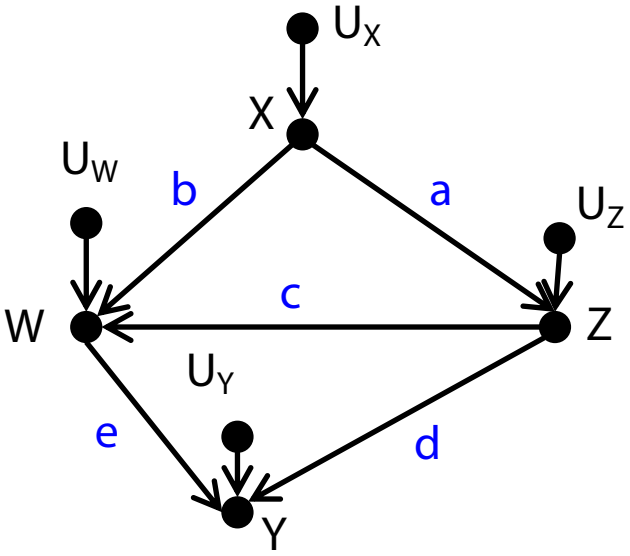
  (Main Reference)

- J. Pearl: Causality, CUP, 2000.

Models with Path Coefficients: SEMs

- Linear SCM
  - $X = U_X$
  - $Z = aX + U_Z$
  - $W = bX + cZ + U_W$
  - $Y = dZ + eW + U_Y$
- Graph with path coefficients
- Path coefficients model Causal Direct Effects (CDEs) w.r.t. change rates
- Last time: U_i = Gaussian error
- But U_i can also be considered as a description of objects

Structural Equational Model (SEM)



If Z is increased by 1 then Y is increase by d

Counterfactuals (Example)

Example (Freeway)

- Came to fork and decided for Sepulveda road (X=0) instead of freeway (X=1)
- Effect: long driving time of 1 hour (Y = 1h)

"If I had taken the freeway,

then I would have driven less than 1 hour"

Counterfactuals (Informal Definition)

Definition

A counterfactual is an if-then statement where:

- the if-condition, aka antecedent, hypothesizes about an alternative non-actual situation/condition (in example: taking freeway) and
- the then-condition, aka succedent, describes some consequence of the hypothetical situation (in example: less than 1h drive)

Counterfactuals ≠ truth-conditional if

- Counterfactuals may be false even if antecedent is false
  - "If Hamburg is capital of Germany,
    then Udo Lindenberg is chancellor" true
  - "If Hamburg had been capital of Germany
    then Udo Lindenberg would have been chancellor" false
- Usually, in natural language use, the antecedent in counterfactuals is false in actual world
- In natural language distinguished by different modes
  - indicative mode for truth-conditional if-statements vs.
  - conjunctive/subjunctive for counterfactuals

Counterfactuals Require Minimal Change

- Hypothetical world minimally different from actual world
  - If X=1 was true (instead of X=0),

    but everything else the same (as far as possible),

    then Y < 1h would be the case

Account for consequences of change (from X= 0 to X = 1).

- Idea of minimal change is ubiquitous
  - See discussion on belief revision in the course "Information Systems"

D. Lewis. Counterfactuals. Harvard University Press, Cambridge, MA, **1973**.

D. Makinson. Five faces of minimality. Studia Logica, 52:339–379, **1993**.

F. Wolter. The algebraic face of minimality. Logic and Logical Philosophy,6:225 – 240, **1998**.

Counterfactuals and Rigidity

- Rigidity as a consequence of minimal change of worlds/states:
  - Objects stay the same in compared worlds

- In example: Driver (characteristics) stays the same:
  - If the driver is a moderate driver, then he will be a moderate driver in the hypothesized world, too

Counterfactuals (Example cont'd)

- **Try:** Formalization with intervention
  - E[driving time |do(freeway), driving time = 1 hour]

    doesn't work! Why?
  - There is a clash for RV „driving time" (Y)
    - Y = 1 h in actual world vs.
    - Y < 1h (expected) under hypothesized condition X =1 (freeway)
- **Solution**: Distinguish Y (driving time) under different worlds/conditions X = 0 vs. X = 1

$$E[Y_{X=1} \mid X = 0, Y_{X=0} = Y = 1]$$

$Y_{X=x}$ formalizes counterfactual

Expected driving time $Y_{X=1}$ if one had chosen freeway (X=1) knowing that other decision (X=0) lead to driving time Y_0 of 1 hour.

Counterfactuals (Definition)

Definition

A counterfactual RV is of the form $Y_{X=x}$ and its semantics is given w.r.t. an instantiation of exogenous variable u by

$$Y_{X=x}(u) := Y_{M_x}(u)$$

Note the rigidity assumption:
Definition talks about the
same "objects" u in different worlds

where

- Y, X are (sets of) RVs from an SEM M
- x is an instantiation of X
- M_x is the SEM resulting from M by substituting the rhs of equation(s) for (all RVs in) X with value(s) x

Counterfactuals (Consistency Rule)

- Consequence of the formal definition of counterfactuals

Consistency rule
If $X = x$, then $Y_{X=x} = Y$

- This case (hypothesized = actual) non-typical in natural language use (Merkel: „If I only would be chancellor...)

Counterfactuals (for Linear SEMs)

- How to formalize semantics of counterfactuals?
  - Use ideas similar to those of intervention
- Consider linear models
  - Values of all variables determined by values of exogenous variables $U = U_1, \ldots, U_n$
  - So can write $X = X(U)$ for any variable in SEM
  - Example
    - X: Salary, $u = u_1, \ldots, u_n$ characterizes individual Joe
    - $X(u)$ = Joe's salary
  - When considering different worlds, the individuals (such as $Joe = (u_1, \ldots, u_n)$) stay the same.

Counterfactuals in linear SEMs (Example)

- Linear model M:

  $$X = aU \quad ; \quad Y = bX + U$$

- Find $Y_{X=x}(u) = ?$
  (value of Y if it were the case that $X = x$ for individual u)

- Algorithm

  1. Identify u under evidence (here: u just given)
  2. Consider modified model M_x
     - $X = x$
     - $Y = bX + U$
  3. Calculate $Y_{X=x}(u)$

     $$Y_{X=x}(u) = bx + u$$

Counterfactuals in linear SEMs (Example)

- Linear model M:

$$X = aU \quad ; \quad Y = bX + U$$

with $a = b = 1$.

$X_y(U) = ?$

Algorithm

1. $U = u$; 2. $Y = y$; 3. $X = aU = au = u$.

(X unaltered by hypothetical condition $Y = y$)

U	X(u)	Y(u)	$Y_{X=1}(u)$	$Y_{X=2}(u)$	$Y_{X=3}(u)$	$X_{Y=1}(u)$	$X_{Y=2}(u)$	$X_{y=3}(u)$
1	1	2	2	3	4	1	1	1
2	2	4	3	4	5	2	2	2
3	3	6	4	5	6	3	3	3

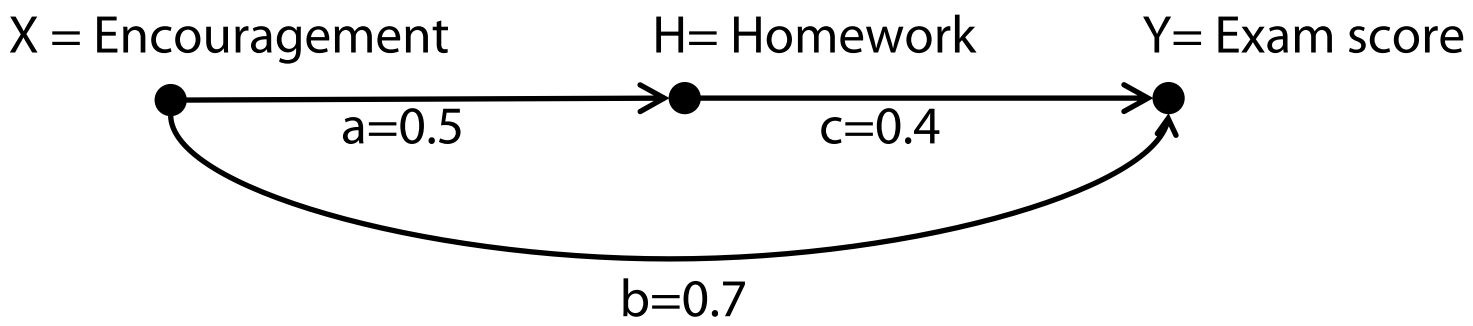
Counterfactuals vs. Intervention with do()

Counterfactual $Y_x(u)$	**Intervention do(X=x)**
Defined locally for each u	Defined globally for whole population/distribution
Can output individual value	Outputs only expectation/distribution
Allows cross-world speak	Allows single-world speak
Can simulate intervention	Cannot simulate counterfactual

Counterfactuals in Linear SEMs (Example)

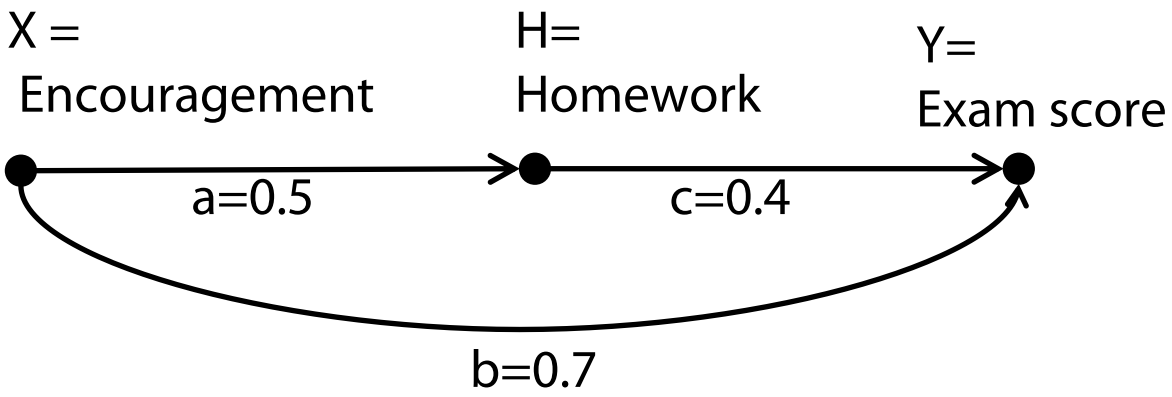
- Linear model M:
  - $X = U_X$
  - $H = aX + U_H$
  - $Y = bX + cH + U_Y$
  - $\sigma_{U_iU_j} = 0$ for all $i,j \in \{X,H,Y\}$ (i.e., U_i, U_j are not linearly correlated/dependent)

$a = 0.5; \quad b = 0.7; \quad c = 0.4$



Counterfactuals in Linear SEMs (Example)

- Linear model M:
  - $X = U_X$
  - $H = aX + U_H$
  - $Y = bX + cH + U_Y$



- Consider an individual Joe given by evidence:

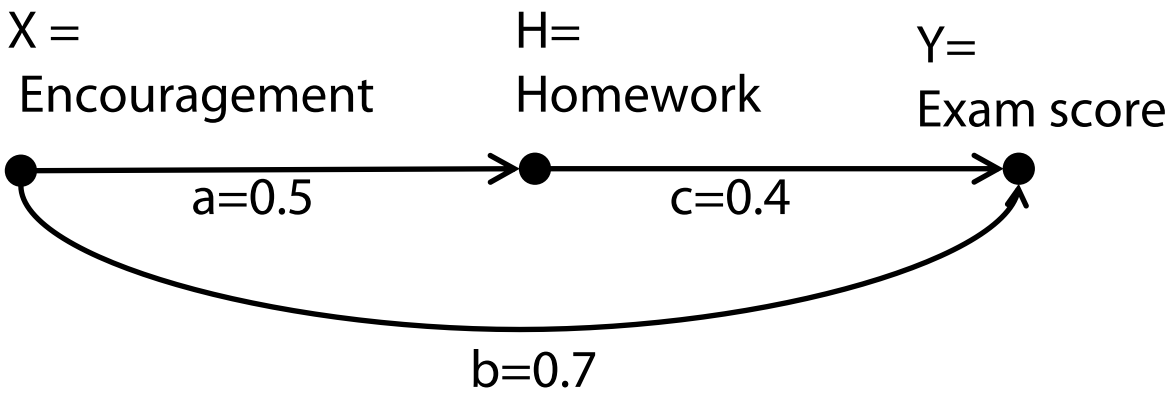
  $X = 0.5, \quad H = 1, \quad Y = 1.5$

- Want to answer counterfactual query:

  „What would have been Joe‘s exam score, if he had doubled study time at home?“

Counterfactuals in Linear SEMs (Example)

- Linear model M:
  - $X = U_X$
  - $H = aX + U_H$
  - $Y = bX + cH + U_Y$



- Consider an individual Joe given by evidence:

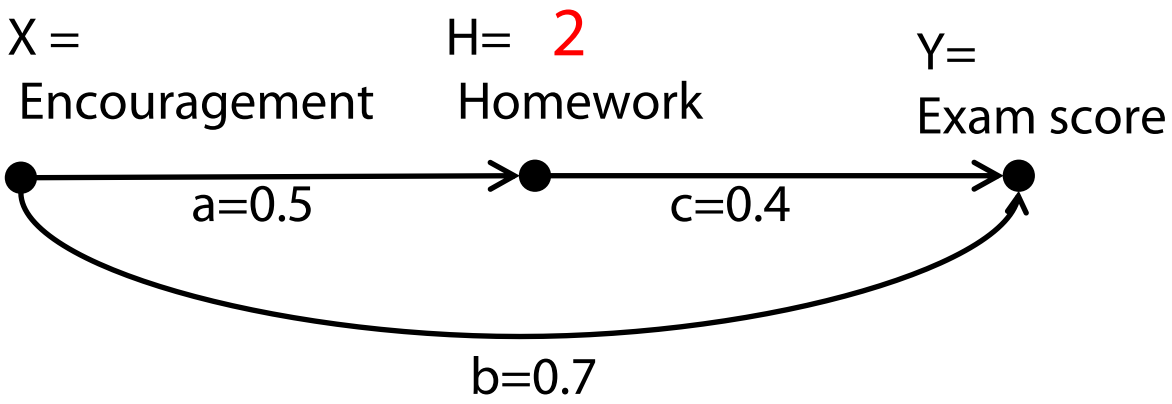
  $X = 0.5, \quad H = 1, \quad Y = 1.5$

- **Step 1**: Determine U-characteristics from evidence
  - $U_X = 0.5$
  - $U_H = 1\text{-}0.5 * 0.5$
  - $U_Y = 1.5 \text{ -}0.7 * 0.5 - 04.4 * 1 = 0.75$

The U-characteristics are rigid

Counterfactuals in Linear SEMs (Example)

- Linear model M:
  - $X = U_X$
  - $H = aX + U_H$
  - $Y = bX + cH + U_Y$



- **Step 2**: Simulate hypothetical change (doubling)
  - Set $H = 2$
- **Step 3:** Calculate counterfactual $Y_{H=2}(u)$
  - $Y_{H=2}(U_X = 0.5, U_h = 0.75, U_Y = 0.75)$
    $= 0.7 * 0.5 + 0.4 * 2 + 0.75 = 1.90$

Joe would benefit from doubling homework

$Y = 1.5$ in actual world, $Y = 1.90$ in hypothetical world when doubling H

Deterministic Counterfactuals Algorithm

Algorithm

- Step 1 (Abduction): Use evidence $E = e$ to determine u
- Step 2 (Action): Modify model M to obtain model M_x
- Step 3 (Prediction): Compute counterfactual $Y_{X=x}(u)$ with M_x

- This algorithm considers single individual
- And answer query is determined by counterfactual value

- What about classes of individuals and probabilistic counterfactuals?

Nondeterministic Counterfactuals Algorithm

Algorithm

- Step 1 (Abduction): Calculate $P(U|E = e)$
- Step 2 (Action): Modify model M to obtain model M_x
- Step 3 (Prediction)**:** Compute expectation $E(Y_{X=x}|E=e)$ using M_x and $P(U|E=e)$

1. Calculate the probabilities of obtaining some individual
2. Same step
3. Calculate conditional expectation: What is the expected value of Y if one were to change X to x knowing $E = e$

Nondeterministic Counterfactuals (Example)

- Model M: $X = aU \;\; ; \;\; Y = bX + U$ (with $a = b = 1$)

  $U = \{1,2,3\}$ represents three types of individuals with prob.

  $P(U = 1) = 1/2; \quad P(U = 2) = 1/3; \quad P(U=3) = 1/6$

- Examples:
  - $P(Y_{X=2} = 3) = ? \qquad = P(U = 1) = 1/2$
  - $P(Y_2 > 3, Y_1 < 4) = \qquad P(U=2) = 1/3$
  - $P(Y_1 < Y_2) = \;\; 1$

U	X(u)	Y(u)	$Y_{X=1}(u)$	$Y_{X=2}(u)$	$Y_{X=3}(u)$	$X_{Y=1}(u)$	$X_{Y=2}(u)$	$X_{y=3}(u)$
1	1	2	2	3	4	1	1	1
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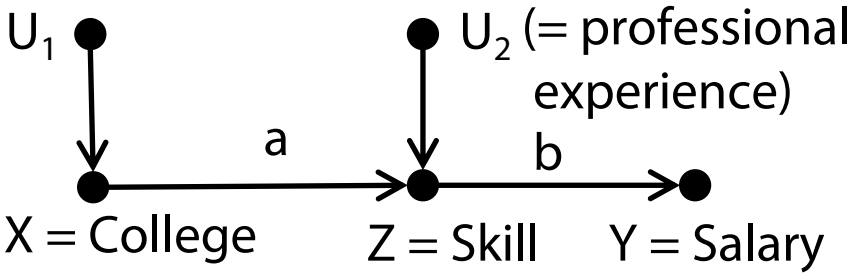
Counterfactuals More Expressive (Example)

- Counterfactuals more expressive than intervention
- Linear model

$X = U_1;\ Z = aX + U_2; Y = bZ$



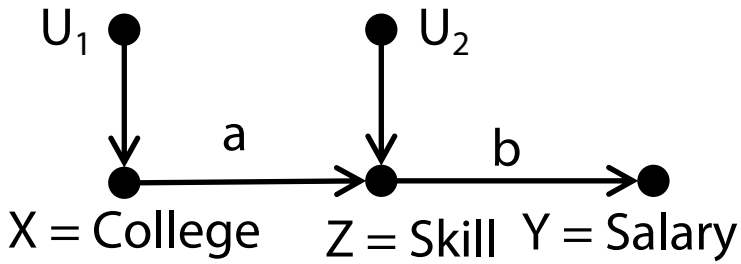
  - $E[Y_{X=1} \mid Z = 1] = ?$
  - Not captured by $E[Y|do(X=1), Z=1]$. Why?
    - Gives only the salary Y of all individuals that went to college and since then acquired skill level Z=1.
    - $E[Y|do(X=1), Z=1] = E[Y|do(X=0), Z=1]$
    - In contrast: $E[Y_{X=1} \mid Z = 1]$ captures salary of individuals who in the actual world have skill level Z =1 but might get Z > 1
    - $E[Y_{X=0} \mid Z = 1] \neq E[Y_{X=1} \mid Z = 1]$

Talks about postintervention for two different groups

Talks about one group acting under different antecedents

Counterfactuals More Expressive (Example)

- $E[Y_{X=0} \mid Z = 1] \neq E[Y_{X=1} \mid Z = 1]$?
  - How is this reflected in numbers?
  - Later: How reflected in graph?



$X = U_1;\ Z = aX + U_2;\ Y = bZ$ (for $a \neq 1$ and $a \neq 0$, $b \neq 0$)

u_1	u_2	X(u)	Z(u)	Y(u)	$Y_{X=0}(u)$	$Y_{X=1}(u)$	$Z_{X=0}(u)$	$Z_{X=1}(u)$
0	0	0	0	0	0	ab	0	a
0	1	0	1	b	b	(a+1)b	1	a+1
1	0	1	a	ab	0	ab	0	a
1	1	1	a+1	(a+1)b	b	(a+1)b	1	a+1

- $E[Y_1|Z=1] = (a+1)b$; $E[Y|do(X=1),Z=1] = b$
- $E[Y_0|Z=1] = b$; $E[Y|do(X=0),Z=1] = b$

In particular: $E[Y_1-Y_0|Z=1] = ab \neq 0$

Counterfactuals vs. Intervention with do()

Counterfactual $Y_x(u)$	**Intervention** do(X=x)
Defined locally for each u	Defined globally for whole population/distribution
Can output individual value	Outputs only expectation/distribution
Allows cross-world speak	Allows single-world speak
Can simulate intervention	Cannot simulate counterfactual

$E[Y|do(X=1), Z=1] = ?$ $= E[Y_{X=1}| Z_{X=1} = 1]$

Counterfactuals vs. Intervention with do()

Counterfactual $Y_x(u)$	**Intervention** do(X=x)
Defined locally for each u	Defined globally for whole population/distribution
Can output individual value	Outputs only expectation/distribution
Allows cross-world speak	Allows single-world speak
Can simulate intervention	Cannot simulate counterfactual

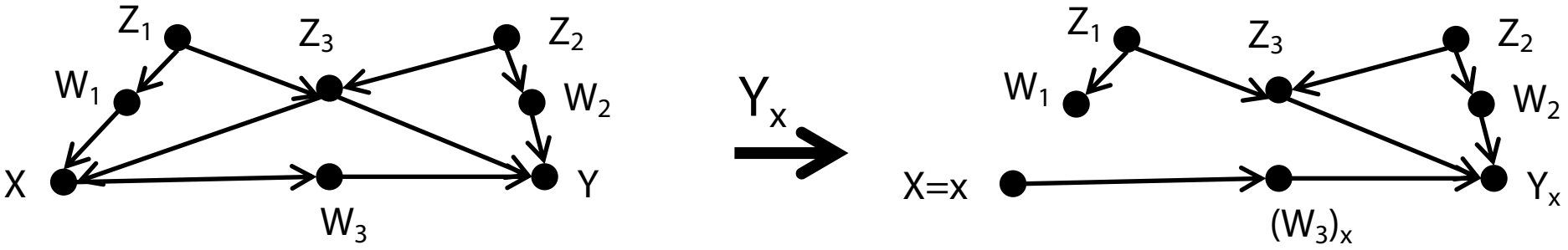
- See road example
- But in non-conditional case we have
  $E[Y_x=y] = E[Y=y|do(X=x)]$

Graphical representation of counterfactuals

- Remember definition of counterfactual

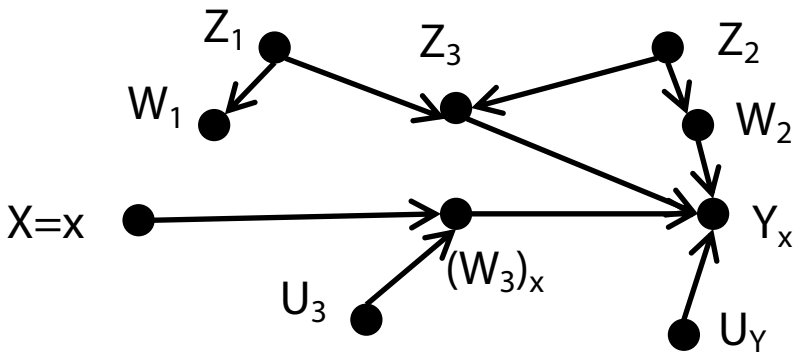
$$Y_{X=x}(u) := Y_{M_x}(u)$$

- Modification as in intervention but with variable change



- Can answer (independence) queries regarding counterfactuals as for any other variable
- Note: Graphs do not show exogenous influences

Independence criterion for counterfactuals



- Which variables can influence Y_x (i.e., Y if X fixed to x)?
  - Parents of Y and parents of nodes on pathway between X and Y (here: $\{Z_3, W_2, U_3, U_y\}$)
- So blocking these with a set of RVs Z renders Y_x independent of X given Z

Theorem (Counterfactual interpretation of backdoor)
If set of RVs Z satisfies backdoor for (X,Y),
then $P(Y_x \mid X,Z) = P(Y_x \mid Z)$ (for all x)

Independence criterion for counterfactuals

Theorem (Counterfactual interpretation of backdoor)
If set of RVs Z satisfies backdoor for (X,Y),
then $P(Y_x \mid X,Z) = P(Y_x \mid Z)$ (for all x)

- Theorem useful for estimating prob. for counterfactuals
- In particular can use adjustment formula

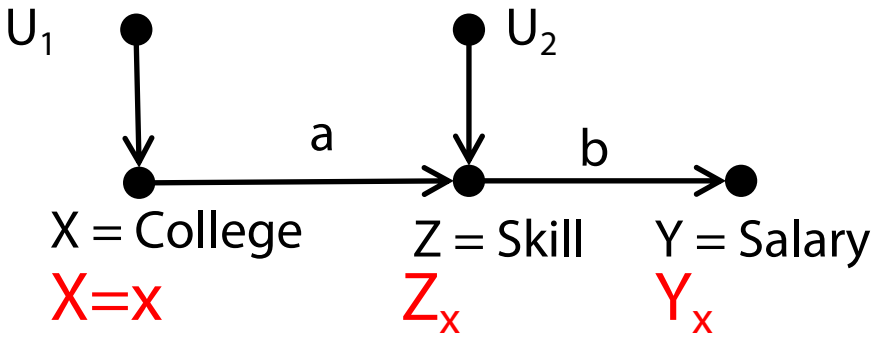
$$P(Y_x = y) = \sum_z P(Y_x = y \mid Z = z)P(z) \quad \text{(summing out)}$$
$$= \sum_z P(Y_x = y \mid Z = z, X=x)P(z) \quad \text{(Thm)}$$
$$= \sum_z P(Y=y \mid Z = z, X = x)\, P(z) \quad \text{(consistency)}$$

- Clear in light of $P(Y_x = y) = P(Y=y \mid do(X=x))$

Independence counterfactuals (example)

- Reconsider linear model

$X = U_1;\ Z = aX + U_2;\ Y = bZ$



- Does college education have effect on salary, considering a group of fixed skill level?
- Formally: Is Y_x independent of X, given Z?
  - Is Y_x d-separated from X given Z?
  - No: Z a collider between X and U_2
  - Hence: $E[Y_x \mid X, Z] \neq E[Y_x \mid Z]$
    (hence education has effect for students of given skill)

Counterfactuals in Linear Models

- In linear models any counterfactual identifiable if linear parameters identified
  - In this case all functions in SEM fully determined
  - Can use $Y_x(u) = Y_{M_x}(u)$ for calculation
- What if some parameters not identified?
  - At least can identify statistical features of form $E[Y_{X=x}|Z=z]$

Theorem (Counterfactual expectation)
Let τ denote slope of total effect of X on Y

$$\tau = E[Y|do(x+1)]-E[Y|do(x)]$$

Then, for any evidence $Z = e$

$$E[Y_{X=x}|Z=e] = E[Y|Z=e] + \tau\,(x-E[X|Z=e])$$

Counterfactuals in Linear Models

Theorem (Counterfactual expectation)

Let τ denote slope of total effect of X on Y

$$\tau = E[Y|do(x+1)]-E[Y|do(x)]$$

Then, for any evidence $Z = e$

$$E[Y_{X=x}|Z=e] = E[Y|Z=e] + \tau\,(x-E[X|Z=e])$$

Current estimate of Y

Expected effect change when x shifted from current best estimate $E[X|Z=e]$

Effect of Treatment on the Treated (ETT)

Theorem (Counterfactual expectation)

Let τ denote slope of total effect of X on Y

$$\tau = E[Y|do(x+1)]-E[Y|do(x)]$$

Then, for any evidence $Z = e$

$$E[Y_{X=x}|Z=e] = E[Y|Z=e] + \tau\,(x-E[X|Z=e])$$

$$\begin{aligned}
ETT &= E[Y_1 - Y_0|X=1] \\
&= E[Y_1|X=1]- E[Y_0|X=1] \\
&= E[Y|X=1]- E[Y|X=1] + \tau\,(1-E[X|X=1]) - \tau\,(0-E[X|X=1]) \\
&\quad \text{(using Thm with } (Z = e) \triangleq (X = 1)\text{)} \\
&= \tau
\end{aligned}$$

Hence, in linear models, effect of treatment on the treated (individual) is the same as total treatment effect on population

Extended Example for ETT

- Job training program (X) for jobless funded by government to increase hiring Y
- Pilot randomized experiment shows:
  Hiring-%(w/ training) > Hiring-%(w/o training) (*)
- Critics
  - (*) not relevant as it might falsely measure effect on those who chose to enroll for program by themselves (these may have gotten job because they are more ambitious)
  - Instead, need to consider ETT
    $E[Y_1 - Y_0 \mid X=1]$ = causal effect of training X on hiring Y for those who took the training

Extended Example for ETT (cont'd)

- Difficult part: $E[Y_{X=0} | X=1]$
  - not given by observational or experimental data
  - but can be reduced to these if appropriate covariates Z (fulfilling backdoor criterion) exist

$P(Y_x = y \mid X = x')$

$= \sum_z P(Y_x = y \mid Z = z, x')P(z|x')$ (by condition on z)

$= \sum_z P(Y_x = y \mid Z = z, x)P(z|x')$ (by Thm on counterfactual backdoor $P(Y_x \mid X,Z) = P(Y_x | Z)$)

$= \sum_z P(Y = y \mid Z = z, x)P(z|x')$ (consistency rule)

Contains only observational/testable RVs

- $E[Y_0|X=1] = \sum_z E(Y \mid Z = z, X=0)P(z|X=1)$

(after substitution and commuting sums)

Extended Example Additive Intervention

- Scenario
  - Add amount q of insulin to group of patients (with different insulin levels)
    - $do(X = X+q) = add_X(q)$
    - Different from simple intervention
  - Calculate effect of additive intervention from data where such additions have not been observed
- Formalization with counterfactual
  - Y = outcome RV = a RV relevant for measuring effect
  - $X = x'$ (previous level of insulin)
  - $Y_{x'+q}$ = outcome after additive intervention with q insulin

Extended Example Additive Intervention

- $E(Y_{x'+q}|x')$ = expected output of additive intervention
  - Part of ETT expression
  - Can be identified with adjustment formula

    (for backdoor Z such as weight, age, etc.)

- $E[Y|add_X(q)] - E[Y]$

  $= \sum_{x'} E[Y_{x'+q}|X=x']P(X=x') - E[Y]$

  $= \sum_{x'}\sum_{z} E[Y|X=x'+q,Z=z]P(Z=z|X=x')P(X=x') - E[Y]$

  (using already derived formula

  $E(Y_x \mid X = x') = \sum_z E(Y = y \mid Z = z, x)P(z|x')$

  and substituting $x = x' + q$)

Extended Example Decision Making (cont'd)

- Scenario 1:
  - Hoping for remission of cancer (Y = 1)
    patient Mrs. Jones has to decide between
    1. Lumpectomy alone (X = 0)
    2. Lumpectomy with irradiation (X = 1)
  - She decides for adding irradiation (X=1)
    and later there is a remission of cancer
  - Is the remission due to her decision?
- Formally: Determine probability of necessity

$$PN = P(Y_{X=0} = 0 \mid X = 1, Y = 1)$$

- If you want remission, you have to go for adding irradiation (irradiation necessary for remission)

Extended Example Decision Making (cont'd)

- Scenario 2
  - Cancer patient Mrs. Smith had lumpectomy alone (X=0) and her tumor reoccurred (Y=0)
  - She regrets not having gone for irradiation
    Is she justified?
- Formally: Determine probability of sufficiency

$$PS = P(Y_{X=1} = 1 \mid X = 0, Y = 0)$$

- If you go for adding irradiation,
  you will achieve cancer remission

Note that, formally, PN and PS are the same.
The distinction comes from interpreting
value 1 = acting
value 0 = omitting an action

Extended Example Decision Making (cont'd)

- Scenario 3
  - Cancer patient Mrs. Daily faces same decision as Mrs. Jones and argues
    - If my tumor is of a type that disappears without irradiation, why should I take irradiation?
    - If my tumor is of a type that does not disappear even with irradiation, why even take irradiation?
  - So, should she go for irradiation?
- Formally:
  Determine probability of necessity and sufficiency

$$PNS = P(Y_{X=1} = 1, Y_{X=0} = 0)$$

Extended Example Decision Making (cont'd)

- Formally: Determine probability of necessity and sufficiency

$$PNS = P(Y_{X=1} = 1, Y_{X=0} = 0)$$

- PN (PS and PNS) can be estimated from data under assumption of monotonicity (adding irradiation cannot cause recurrence of tumor)

$$PNS = P(Y=1|do(X=1)) - P(Y=1|do(X=0))$$

= total effect on Y of changing X from no irradiation to irradiation

Summary

- Counterfactual reasoning is not intervention
  - Can simulate intervention
- Counterfactual reasoning required for certain applications
  - Compute the effect of different options
  - Reason about nessecity and sufficiency of diagnoses
- Can do counterfactual reasoning in some cases even if models are incomplete