Tech Reports

# Nearness Rules and Scaled Proximity (Extended Version)

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**Abstract**

An artificial intelligence system that processes geo-thematic data would profit from a (semi-)formal or controlled natural language interface that incorporates concepts for nearness. Though there already exists logical-engineering approaches giving sufficient conditions for nearness relations, we show within a logical analysis that these suffer from some deficiencies. Non-engineering approaches to nearness such as the abstract mathematical approach based on proximity spaces do not deal with the implementation aspects but axiomatically formalize intuitions on nearness relations and provide insights on their nature. Combining the ideas of the engineering approach with the mathematical approach of proximity spaces, we define and analyze new nearness relations that provide a good compromise between implementation needs and the need for an appropriate approximation of the natural nearness concept.

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# 1 Introduction

In many use cases the interaction with an AI system for processing geo-thematic data (e.g., robotics, geographical information systems) is conducted by querying it through a specific query language. Most of these systems either do not have a query language that deserves the name as it lacks a declarative specification with a precise logical semantics for spatial concepts; or the query language is so complex that it can be used only by experts. The (old) appealing idea of interacting through a natural language still lacks a satisfying realization due to the hard problems involved in natural language processing. So the idea of combining the flexibility and affinity of natural language for human users with the precise semantics of formal languages in a semi-formal or controlled natural language (CNL) seems to be a good compromise for an adequate interface to systems that process geo-thematic data.

The work of Grütter and colleagues [8, 7] can be understood as a first step towards a CNL that has the capacities to represent and process spatial queries. They focus on a qualitative model of a nearness (closeness) relation which they base on administrative and functional regions. The investigated nearness relation is defined in the logico-formal framework of the region connection calculus [12] equipped with a hierarchical structure of partitions, and is intended to approximate the natural nearness concept used by humans.

As the nearness relation (as any other concept of a CNL) can only be an approximation of the corresponding natural concept, one has to inform the user of the CNL about the properties of the defined concepts. In particular, the user should get a clear picture of which properties the nearness relation of the CNL has. Experimental investigations as those conducted in the articles [8, 7] are a first step towards understanding the nearness relations; but these alone do not give a complete picture needed to justify the specific models of nearness and the CNL in which they are (going to be) embedded.

In this paper, we fill the gap by providing a logical analysis of the nearness relations of [8, 7]. We show that the nearness relation in the original definition of [8, 7] has some desirable properties which a user would expect to be owned by a nearness relation. But it has also some properties that a user would not expect to be shared by nearness relations. Some of the deficiencies of the old nearness definitions can be overcome—and we will do so by giving new definitions. But some of the properties of the nearness relations are inherently associated with the hierarchical approach—and make it essentially different from the usual nearness concepts.

This difference is demonstrated within the well investigated mathematical framework of proximity spaces; these formally axiomatize nearness relations [9]—but are not constructed w.r.t. to the implementation aspects. The result of the proximity-spaces oriented analysis is that nearness relations do not fulfill the axioms for proximity spaces because the nearness relations are not only inherently not symmetric but also depend on the hierarchical context given by the second argument. But we can show that the (extended) nearness relations fulfill some weakening of the proximity axioms. This result has the consequence that there are more general structures than proximity spaces which are worth to be investigated mathematically because they are exemplified by the nearness relations defined in this paper.

The main contribution of this paper is that we logically analyze (variants) of the nearness relations of [8, 7] and then—based on the analysis—define a new nearness relation which combines the ideas of the (construction oriented) engineering approach, that is based on scaling context determining partitions, and the (abstract, axiomatic) mathematical approach of proximity spaces, that axiomatically specifies the properties of nearness relations. Thereby we provide a candidate component for a (semi)-formal language or a CNL that can be used as an interface for AI systems processing geo-thematic data.

The paper is structured as follows. Section 2 gathers the concepts of the region connection calculus [12] needed to define the nearness relations. The following Section 3 recapitulates the definitions of the nearness relations of [8, 7]. In Sections 4 and 5 we analyze the logical properties of the nearness relations—incorporating the general axioms of proximity spaces. Resulting from the analysis of these sections, in the section before the conclusion (Sect. 6) we extend and modify the nearness relations in order to cope with some deficiencies of the old definitions.

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# 2 Preliminaries

The nearness rules of [8, 7] are defined among others with the help of binary relations of the region connection calculus (RCC) [12]—thereby extending RCC with a concept of nearness that is different from the mereotopological nearness concept associated with the connectedness relation. RCC is a family of calculi for qualitative spatial reasoning which is built on regions and not points as basic entities. Starting with a binary reflexive and symmetric connectedness relation $\mathsf{C}$, different binary relations are defined. So the axiomatization of RCC according to [12] is based on the following axioms stating the symmetry and reflexivity of $\mathsf{C}$:

$$\forall x.\mathsf{C}(x,x) \tag{1}$$

$$\forall x,y[\mathsf{C}(x,y) \rightarrow \mathsf{C}(y,x)] \tag{2}$$

The family of calculi RCCi (for $i \in \{1,2,3,5,8\}$) are characterized by sets $\mathcal{B}_{RCCi}$ of $i$ *base relations* $\mathcal{B}_{RCCi} = \{r_1,\ldots,r_i\}$ which have the JEPD-property: they are jointly exhaustive and mutually disjoint, i.e., for all $x,y$ does $r_i(x,y)$ hold exactly for one $r_i$. More general relations are constructed by disjunctions of the base relations. We will work here with the relations of the most expressive calculus RCC8 which rests on the set of base relations $\mathcal{B}_{RCC8} = \{\mathsf{DC}, \mathsf{EC}, \mathsf{EQ}, \mathsf{PO}, \mathsf{NTPP}, \mathsf{TPP}, \mathsf{NTPPi}, \mathsf{TPPi}\}$. The definitions of the relations of $\mathcal{B}_{RCC8}$ as well as other relations we will be using in the following are given as predicate logical sentences in the following list (see, e.g., [13, p. 42]). The meanings of the RCC8 base relations are illustrated in Fig. 1 for two discs $x,y$ in the plane $\mathbb{R} \times \mathbb{R}$.

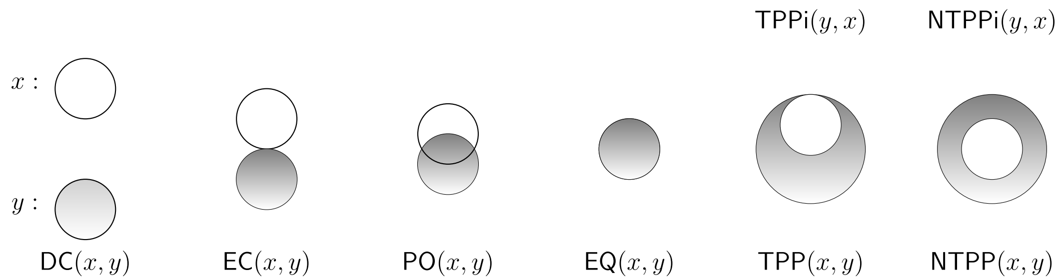


Figure 1: Base relations of RCC8 and their intended meanings

$$\mathsf{DC}(x,y) \leftrightarrow \neg\mathsf{C}(x,y)\text{(disconnnected)} \tag{3}$$

$$\mathsf{P}(x,y) \leftrightarrow \forall z(\mathsf{C}(z,x) \rightarrow \mathsf{C}(z,y))\text{(part of)} \tag{4}$$

$$\mathsf{PP}(x,y) \leftrightarrow \mathsf{P}(x,y) \wedge \neg\mathsf{P}(y,x)\text{(proper part of)} \tag{5}$$

$$\mathsf{EQ}(x,y) \leftrightarrow \mathsf{P}(x,y) \wedge \mathsf{P}(y,x)\text{(equal)} \tag{6}$$

$$\mathsf{O}(x,y) \leftrightarrow \exists z(\mathsf{P}(z,x) \wedge \mathsf{P}(z,y))\text{(overlap)} \tag{7}$$

$$\mathsf{PO}(x,y) \leftrightarrow \mathsf{O}(x,y) \wedge \neg\mathsf{P}(x,y) \wedge \neg\mathsf{P}(y,x)\text{(partial overlap)} \tag{8}$$

$$\mathsf{DR}(x,y) \leftrightarrow \neg\mathsf{O}(x,y)\text{(discrete)} \tag{9}$$

$$\mathsf{EC}(x,y) \leftrightarrow \mathsf{C}(x,y) \wedge \neg\mathsf{O}(x,y)\text{(externally connected)} \tag{10}$$

$$\mathsf{TPP}(x,y) \leftrightarrow \mathsf{PP}(x,y) \wedge \exists z(\mathsf{EC}(z,x) \wedge \mathsf{EC}(z,y))\text{(tangential proper part)} \tag{11}$$

$$\mathsf{NTPP}(x,y) \leftrightarrow \mathsf{PP}(x,y) \wedge \neg\exists z(\mathsf{EC}(z,x) \wedge \mathsf{EC}(z,y))\text{(non-tangential proper part)} \tag{12}$$

$$\mathsf{TPPi}(x,y) \leftrightarrow \mathsf{TPP}(y,x)\text{(inverse of tangential proper part)} \tag{13}$$

$$\mathsf{NTPPi}(x,y) \leftrightarrow \mathsf{NTPP}(y,x)\text{(inverse of non-tangential proper part).} \tag{14}$$

Note that for two regions $x, y$, $\mathsf{EQ}(x,y)$ means that $x, y$ cover the same area in space. RCC8 allows for models in which $\mathsf{EQ}(x,y)$ may hold even if $x \neq y$, that is even if $x, y$ denote different objects. This distinction is also useful for a detailed representation of administrative regions as referenced in the framework of [7] which we adopt in this paper. For example, a community is strictly speaking not just a spatially extended object but some "abstract" entity with specific political obligations, functions etc. But in this paper we will not differentiate between objects and their local extensions as on the one hand we want to keep the logical framework for which we want to give a logical analysis as simple as possible; and on the other hand we will not deal with further thematic axiomatizations in form of complex ontologies for which a distinction between objects and their local extensions would even be necessary. Hence, though we will continue to use the symbol $\mathsf{EQ}$ we will assume that it can be substituted by the identity $=$. Using a term of the spatial region literature we will consider only *strict models* of RCC8 [15]. But note that all results (with slight adaptations) also hold for non-strict models of RCC8.

Beside the definitions of the base relations the axiom system of Randell and colleagues contains the axiom of non-atomicity which states that every region has a non-tangential proper part—which immediately leads to an infinite set of regions.

$$\forall x \exists y.\mathsf{NTPP}(y,x) \tag{15}$$

This axiom will be used in a proof as a technical aid. Though non-atomicity leads to infinite models, the numbers of partitions and level determining regions constituting the partitions (see below) will be finite. Randell and colleagues also define binary functions for regions; one of these is the sum function for regions $x, y$ which results in the union $z$ of $x, y$. The definition is given by the following axiom:

$$\forall x, y, z[\mathsf{sum}(x, y) = z \leftrightarrow \forall w(\mathsf{C}(w, z) \leftrightarrow (\mathsf{C}(w, x) \vee \mathsf{C}(w, y)))] \tag{16}$$

That means, $z$ is the sum of $x$ and $y$ if and only if any region $w$ connects to the sum iff it connects to one of the summands. Instead of $\mathsf{sum}(x, y)$ we also use the set theoretic notation $x \cup y$ and we assume that the sum function is extended to any finite number of arguments (using the associativity of $\mathsf{sum}$) so that also $\bigcup_{i \in I} x_i$ for any finite index set $I$ is defined. The other boolean functions are those for intersection, complement and difference. Complementation is defined by:

$$\forall x, y. \mathsf{compl}(x) = y \leftrightarrow \forall z[(\mathsf{C}(z, y) \leftrightarrow \neg\mathsf{NTPP}(z, x)) \wedge (\mathsf{O}(z, y) \leftrightarrow \neg\mathsf{P}(z, x))] \tag{17}$$

The intersection or product $\mathsf{prod}(x, y)$ (also denoted set theoretically by $x \cap y$) is defined by the following axiom

$$\mathsf{prod}(x, y) = z \leftrightarrow \forall u[\mathsf{C}(u, z) \leftrightarrow \exists v(\mathsf{P}(v, x) \wedge \mathsf{P}(v, y) \wedge \mathsf{C}(u, v))] \tag{18}$$

And the difference $\mathsf{diff}(x, y)$—also denoted set theoretically by $x \setminus y$—is defined as follows:

$$\mathsf{diff}(x, y) = w \leftrightarrow \forall z[\mathsf{C}(z, w) \leftrightarrow \mathsf{C}(z, \mathsf{prod}(x, \mathsf{compl}(y)))] \tag{19}$$

We call the set $Ax_{BRCC}$ consisting exactly of the axioms in (1)–(19) *the axiom set for the boolean region connection calculus* and denote it by BRCC. A model of BRCC is given by interpreting regions as regular closed subset of $\mathbb{R}^2$ equipped with the usual topology. In this model, regular closed sets $x, y$ are connected iff they share a point, i.e., $\mathsf{C}(x, y)$ iff $x \cap y \neq \emptyset$.

# 3 A-priori and General Nearness

Let be given a region $X$. A partition $(a_i)_{i \in I}$ over $X$ is a family of sets $a_i$ such that $\bigcup_{i \in I} a_i = X$ and the $a_i$ are pairwise discrete, i.e., if $i \neq j$ then $a_i \{\mathsf{DC}, \mathsf{EC}\} a_j$. The partition is called finite if the index set $I$ is finite. Note that this is not exactly the same notion of partition as in set theory, as we de not have $a_i \cap a_j = \emptyset$. This is due to the fact that we want to use RCC8 and think of the $a_i$ as regularly closed subsets of $\mathbb{R}^2$ which have borderlines. In all of our examples we will think of $X$ and the $a_i$ as regular closed subsets of $\mathbb{R}^2$.

Given $X$ we consider not one finite partition over $X$ but a finite number of partitions; these are totally ordered whereby a partition is smaller than another partition if and only if the former is finer than the latter. This can be formalized by the relation $\leq$ between partitions $(a_i)_{i \in I}$ and $(b_i)_{i \in I'}$ as follows: $(a_i)_{i \in I} \leq (b_i)_{i \in I'}$ iff every $b_i$ is the union of some $a_j$. Now we fix a finite set $J = \{1, \ldots, n\}$ for the $n$ partitions over $X$. A partition $(a_i^j)_{i \in I_j}$ of these finite number of partitions has a superscript $j \in J$ for the position it has in the total ordering of the partitions w.r.t. $\leq$. The finite index set $I_j$ has the cardinality of the number

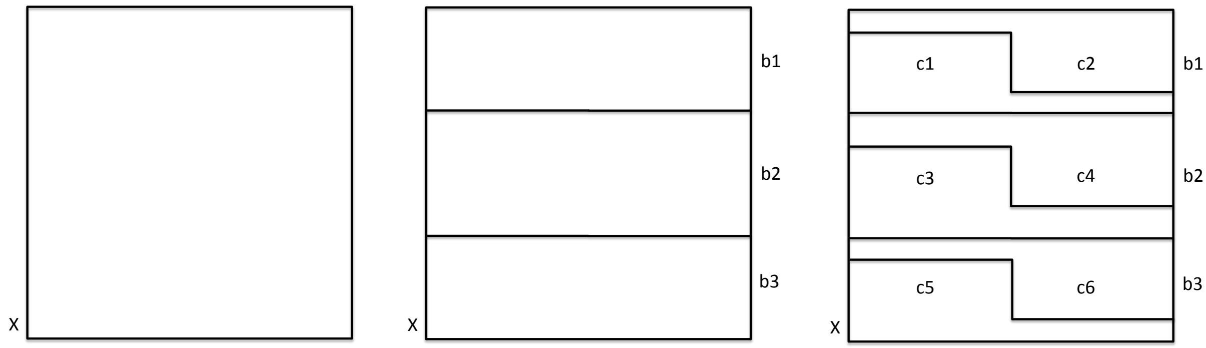


Figure 2: A total order of three partitions: $(c_i)_{i\in\{1,2,3,4,5,6\}} \leq (b_i)_{i\in\{1,2,3\}} \leq (X)$

of cells constituting the partition $(a_i^j)_{i\in I_j}$. We also say that the partition and all the cells in it are of level $j$. So the fixed set of totally ordered partitions is given as:

$$(a_i^1)_{i\in I_1} \leq (a_i^2)_{i\in I_2} \leq \cdots \leq (a_i^n)_{i\in I_n} \tag{20}$$

We assume that the last partition $(a_i^n)_{i\in I_n}$ in the order consists of just one set, namely the set $X$. (Moreover, as we work here with strict RCC models, we assume that no region occurs in two different partitions. This assumption can be dropped for non-strict models.)

For every partition $(a_i^j)_{i\in I_j}$ with $j < n$ there is the partition $(a_i^{j+1})_{i\in I_{j+1}}$ directly following it in the total ordering $\leq$; we call it the *partition of the next (upper) level.* As in some cases the exact instances of the index sets and superscripts are not relevant, we sometimes neglect them and, e.g., just write $(a)$ for $(a_i^j)_{i\in I_j}$. Then the partition following $(a)$ is denoted conveniently by $(a)^\uparrow$. We call all the regions occurring as a cell in one of the partitions a *level-determining set/region, ld set/ld region* for short. In the work of [8] the administrative regions have the role of level determining regions.

An example for a total order of partitions is given in Fig. 2. In this example we have three partitions ($n = 3$); the partition of the "highest" level which is the coarsest level is denoted by $(X)$. Using the full indexed notation from above we would denote $(X)$ by $(a_i^3)_{i\in\{1\}}$. The next lower level is given by the partition $(b) = (b_i)_{i\in\{1,2,3\}}$; using the full index notation it would be written as $(b) = (a_i^2)_{i\in\{1,2,3\}}$. We have $(b)^\uparrow = (X)$; and the last partition (the finest one) is given by $(c) = (c_i)_{i\in\{1,2,3,4,5,6\}}$ written in the full indexed version as $(c) = (a_i^1)_{i\in\{1,2,3,4,5,6\}}$. We have $(c)^\uparrow = (b)$.

Presupposing a finite total order of partitions that have a finite number of cells means that we work within a closed world of ld regions; hence an all-quantified assertion of the form $\forall x_a \in (a_i)_{i\in\{1,\ldots,n\}} P(x_a)$ for a unary predicate $P$ and a partition $(a_i)_{i\in\{1,\ldots,n\}}$ can be equivalently written by the finite conjunction $\bigwedge_{i\in\{1,\ldots,n\}} P(a_i)$. Similarly $\exists x_a \in (a_i)_{i\in\{1,\ldots,n\}} P(x_a)$ is written as a finite disjunction $\bigvee_{i\in\{1,\ldots,n\}} P(a_i)$. So all rules below can be transformed to quantifier free versions and hence actually described in a propositional logical framework.

In all of the following definitions let $(a) = (a_i)_{i\in I}$ denote a finite partition of the fixed (finite) total order of partitions. The nearness relation of [7] is based on a notion of apriori nearness between administrative regions. Their notion of apriori nearness also incorporates a reference to functional regions which is justified by the observation that the nearness perception by cognitive agents is influenced by the borders of functional regions. In the following considerations we will drop this condition incorporating functional regions, hence use the following simpler rule for a-priori nearness $\mathsf{NR_{ap}}$ between ld regions of the same level:

$$\forall x_a \in (a) \forall y_a \in (a) \forall b \in (a)^{\uparrow}[(x_a\{\mathsf{P}\}b \wedge y_a\{\mathsf{P},\mathsf{EC}\}b) \rightarrow \mathsf{NR_{ap}}(y_a, x_a)] \tag{21}$$

Moreover, we set $\mathsf{NR_{ap}}(X,X)$, so that reflexivity holds for $\mathsf{NR_{ap}}$ (see below).

$$\mathsf{NR_{ap}}(X,X) \tag{22}$$

So, according to the definition, an ld region $y_a$ is a-priori near $x_a$ if $y_a$ touches or is contained in the next upper level cell $b$ of which the ld region $x_a$ is a part. Hence, the second argument $x_a$ determines the scaling or granularity or level context for the nearness comparison.

The more general nearness relation $\mathsf{NR}$ may hold between regions of different partitions. Furthermore, in the sufficient condition stated below the variable $z$ may denote any region—not necessarily an ld region.

$$\forall x_a \in (a) \forall y_a \in (a) \forall z[(z\{\mathsf{P},\mathsf{PO}\}y_a \wedge \mathsf{NR_{ap}}(y_a, x_a)) \rightarrow \mathsf{NR}(z, x_a)] \tag{23}$$

In order to investigate the properties of these nearness relations we will assume that the fixed totally ordered partition is represented by a finite set $\mathcal{A}$ of predicate logical axioms. If for example, an ordering is given as in Fig. 2, then $\mathcal{A}$ would consist of the following axioms, where $X, a_1, \ldots, a_6, b_1, b_2, b_3$ denote constants and $ld$ is a unary predicate symbol intended to denote ld regions.

$$X = b_1 \cup b_2 \cup b_3 \wedge b_1\{\mathsf{EC}\}b_2 \wedge b_1\{\mathsf{DC}\}b_3 \wedge b_2\{\mathsf{EC}\}b_3 \tag{24}$$

$$b_1 = c_1 \cup c_2 \wedge b_2 = c_3 \cup c_4 \wedge b_3 = c_4 \cup c_5 \tag{25}$$

$$c_1\{\mathsf{EC}\}c_2 \wedge c_1\{\mathsf{EC}\}c_4 \wedge c_3\{\mathsf{EC}\}c_4 \wedge c_3\{\mathsf{EC}\}c_6 \wedge c_6\{\mathsf{EC}\}c_5 \tag{26}$$

$$c_2\{\mathsf{DC}\}c_3 \wedge c_2\{\mathsf{DC}\}c_4 \wedge c_2\{\mathsf{DC}\}c_5 \tag{27}$$

$$c_2\{\mathsf{DC}\}c_6 \wedge c_1\{\mathsf{DC}\}c_3 \wedge c_1\{\mathsf{DC}\}c_5 \tag{28}$$

$$c_1\{\mathsf{DC}\}c_6 \wedge c_4\{\mathsf{DC}\}c_5 \wedge c_4\{\mathsf{DC}\}c_6 \wedge c_3\{\mathsf{DC}\}c_5 \tag{29}$$

$$\forall x[ld(x) \leftrightarrow (x = X \vee x = b_1 \vee x = b_2 \vee x = b_3 \vee x = c_1 \vee \cdots \vee x = c_6)] \tag{30}$$

Moreover, we assume that an axiomatization $Ax_{BRCC8}$ of BRCC8 is given as defined in the preliminaries. Now, let $\mathcal{KB} = \mathcal{A} \cup Ax_{BRCC8} \cup \{(21), (22), (23)\}$ be a knowledge base consisting of the closed-world axioms for the total order of partitions plus the axioms for the boolean region connection calculus plus the rules for a-priori nearness (21), (22) and the rule (23) for (general) nearness. The investigations of the logical properties of the nearness relations are done with respect to this knowledge base $\mathcal{KB}$.

# 4 Properties of the Nearness Relations

We start our analysis with some simple observations concerning the a-priori nearness relation $\mathsf{NR_{ap}}$, its relation to $\mathsf{NR}$ and move on to the analysis of the properties of $\mathsf{NR}$.

A-priori nearness can hold only between ld regions of the same level. That means, if two (different) regions are derived to be near a-priori, then they are either disjoint ($\mathsf{DC}$) or touch each other ($\mathsf{EC}$). Concerning the converse of this fact we can observe that two touching regions of the same partition are a priori near. Clearly, this is a desirable feature of a-priori nearness as it shows that a-priori nearness is compatible with the mereotopological nearness relation of two touching regions.

**Proposition 1.** *If $y_a$ and $x_a$ are regions of the same partition $(a)$ and $\mathcal{KB} \models x_a\{\mathsf{EC}\}y_a$, then $\mathcal{KB} \models \mathsf{NR}_{\mathsf{ap}}(x_a, y_a)$ and $\mathcal{KB} \models \mathsf{NR}_{\mathsf{ap}}(y_a, x_a)$.*

*Proof.* Let $b_x$ be the region of the next upper partition such that $\mathcal{KB} \models x_a\{\mathsf{EQ}, \mathsf{TPP}, \mathsf{NTPP}\}b_x$ and $b_y$ be the region of the next upper partition such that $\mathcal{KB}$ implies $y_a\{\mathsf{EQ}, \mathsf{TPP}, \mathsf{NTPP}\}b_y$. If $b_x\{\mathsf{EQ}\}b_y$, i.e., $b := b_x = b_y$, then $x_a\{\mathsf{P}\}b$ and $y_a\{\mathsf{P}\}b$, hence $\mathsf{NR}_{\mathsf{ap}}(x_a, y_a)$ and $\mathsf{NR}_{\mathsf{ap}}(y_a, x_a)$. Otherwise, together with $x_a\{\mathsf{EC}\}y_a$ and $b_x\{\mathsf{EC}, \mathsf{DC}\}b_y$ it follows that $\mathcal{KB}$ implies $x_a\{\mathsf{EC}\}b_y$ and $y_a\{\mathsf{EC}\}b_x$, so $\mathsf{NR}_{\mathsf{ap}}(x_a, y_a)$ and $\mathsf{NR}_{\mathsf{ap}}(y_a, x_a)$. □

Another desired feature of a-priori nearness $\mathsf{NR}_{\mathsf{ap}}$ is reflexivity because a region should count near itself—it can be even thought of as being one of the regions that are nearest itself. Indeed, reflexivity (restricted to ld regions) holds for $\mathsf{NR}_{\mathsf{ap}}$.

**Proposition 2.** *For all $x_a \in (a_i)_{i \in I}$: $\mathcal{KB} \models \mathsf{NR}_{\mathsf{ap}}(x_a, x_a)$.*

*Proof.* If $x_a = X$, this follows directly from the definition (see (22)). Otherwise $x_a \neq X$ and there is $b \in (a)^{\uparrow}$ such that $x_a\{\mathsf{P}\}b$. Now, as $x_a\{\mathsf{EQ}\}x_a$, one has also $x_a\{\mathsf{P}, \mathsf{EC}\}x_a$, that is $x_a\{\mathsf{P}, \mathsf{EC}\}b$. □

Clearly $\mathsf{NR}_{\mathsf{ap}}$ is not symmetric w.r.t. to $\mathcal{KB}$. By this we mean that the following symmetry condition does not hold: If $\mathcal{KB} \models \mathsf{NR}_{\mathsf{ap}}(x, y)$, then $\mathcal{KB} \models \mathsf{NR}_{\mathsf{ap}}(y, x)$. And clearly $\mathsf{NR}_{\mathsf{ap}}$ is not transitive w.r.t. $\mathcal{KB}$, i.e., the following transitivity condition does not hold: If we have $\mathcal{KB} \models \mathsf{NR}_{\mathsf{ap}}(x, y)$ and $\mathcal{KB} \models \mathsf{NR}_{\mathsf{ap}}(y, z)$, then $\mathcal{KB} \models \mathsf{NR}_{\mathsf{ap}}(x, z)$. These facts are demonstrated with the example given in Fig. 2. It holds that $\mathsf{NR}_{\mathsf{ap}}(c_6, c_4)$ as verified by the region $b_2$ for which we have $c_4\{\mathsf{P}\}b_2$ and $c_6\{\mathsf{EC}\}b_2$. But the symmetric relation $\mathsf{NR}_{\mathsf{ap}}(c_4, c_6)$ does not hold, because not $c_4\{\mathsf{EC}\}b_3$. Similarly one can see that $\mathsf{NR}_{\mathsf{ap}}(c_4, c_2)$; but $\mathsf{NR}_{\mathsf{ap}}(c_6, c_2)$ does not hold which shows that transitivity is not given.

The general nearness relation extends the apriori nearness relation in a conservative manner. That is, any two regions which are apriori-near are near as well. Moreover, if the regions are ld regions of the same level, then the converse holds as well.

**Proposition 3.** *For all $y_a, x_a \in (a)$: $\mathcal{KB}$ implies $\mathsf{NR}_{\mathsf{ap}}(y_a, x_a)$ iff it implies $\mathsf{NR}(y_a, x_a)$.*

*Proof.* The direction from left to right follows directly from the definitions and the fact that $y_a\{\mathsf{P}, \mathsf{PO}\}y_a$. If on the other hand $\mathsf{NR}(y_a, x_a)$, then there is a $y'$ of the same level as $x_a$ such that $\mathsf{NR}_{\mathsf{ap}}(y', x_a)$ and $y_a\{\mathsf{P}, \mathsf{PO}\}y'$. But as $y_a, y'$ have the same level, only $y_a\{\mathsf{EQ}\}x_a$ can hold, and hence $\mathsf{NR}_{\mathsf{ap}}(y_a, x_a)$. □

Reflexivity does not hold in case of $\mathsf{NR}$, as $\mathsf{NR}$ allows only for ld regions as second arguments. This is an unwanted feature for nearness relations which we will deal with in one of the following sections (page 6) by extending the nearness relation to a new nearness relation. For this extension we will use the simple fact, that if $z$ is an ld region, then reflexivity holds. (This is a direct consequence of Prop. 3.)

**Proposition 4.** *For all ld regions $x_a$: $\mathcal{KB} \models \mathsf{NR}(x_a, x_a)$.*

As in the case of a-priori nearness we can immediately see that $\mathsf{NR}$ is not symmetric and not transitive.[1] These facts can be explained again by the fact that it is the second argument which determines the comparison context.

For a-priori nearness $\mathsf{NR}_{\mathsf{ap}}$ we answered the question which base RCC8 relations $r$ are sufficient for nearness, i.e., for which $r \in \mathcal{B}_{RCC8}$ does $r(z,x)$ imply $\mathsf{NR}_{\mathsf{ap}}(z,x)$. Lifting this question to general nearness becomes more interesting as there are more possible RCC8 relations between a region $z$ and an ld region $x_a$. Unfortunately, for $r = \mathsf{EC}$ the entailment $r(z,x_a) \models \mathsf{NR}(z,x_a)$ does not hold. Take a region $z$ that touches $x_a$ but is neither contained nor overlaps an a-priori near region $y_a$ of the same partition as $x_a$; e.g., if $(a_i)_{i\in I}$ is the partition of $x_a$ consider the region $z = (\bigcup_{i\in I} a_i) \setminus x_a$. (Note that we use here the boolean operator of difference $\setminus$).

Similarly one can show that $z\{\mathsf{TPPi}, \mathsf{NTPPi}\}x_a$ does not entail $\mathsf{NR}(z,x_a)$. This seems to be an implausible property of the $\mathsf{NR}$-definition. One may argue that this consequence is due to the scaling dependence of $\mathsf{NR}$ on the second argument: The second argument of $\mathsf{NR}$ defines the context, the granularity or the scale with respect to which nearness is considered; if $z\{\mathsf{TPPi}, \mathsf{NTPPi}\}x_a$, then—one might argue that—$z$ is "too big" for the scaling context given by the second argument $x_a$; hence one may conclude that considering the nearness between $z$ and $x_a$ is not even justified from the beginning. But similarly one could argue that the second argument $x_a$ determines the scaling in the sense that more regions will be detected as near $x_a$ than will w.r.t. to region $y_{a^\uparrow}$ of an upper level partition $(a)^\uparrow$. Hence, we will later on (see page 6) look at a more general notion of nearness that also allows for "big" regions $z$ being near a region $x_a$.

Excluding the above cases for $r$ results in the set $\{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}, \mathsf{PO}\}$ of possible instances for $r$. And as the following proposition shows, if regions $z, x_a$ stand in one of the base relations in $\{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}, \mathsf{PO}\}$, then nearness is guaranteed.

**Proposition 5.** *For all $z, x_a$: If $\mathcal{KB} \models z\{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}, \mathsf{PO}\}x_a$, then $\mathcal{KB} \models \mathsf{NR}(z,x_a)$.*

*Proof.* Let $\mathcal{KB} \models z\{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}, \mathsf{PO}\}x_a$. That means $\mathcal{KB} \models z\{\mathsf{P}, \mathsf{PO}\}x_a$; hence with Prop. 2 it follows that $\mathsf{NR}_{\mathsf{ap}}(x_a,x_a)$ and $\mathsf{NR}(z,x_a)$. □

As a corollary of this proposition and the definition of $\mathsf{NR}$ we note that all ld regions are in $\mathsf{NR}$-relation to ld regions (of upper levels) of which they are a part.

As a last observation, we note that $\mathsf{NR}(z,x_a)$ is compatible with (or independent of) all base relations of RCC8 in the following sense: one can find for any $r \in \mathcal{B}_{RCC8}$ regions $z$ and $x_a$ such that $\mathcal{KB} \models \mathsf{NR}(z,x_a) \wedge r(z,x_a)$. Hence, if one knows that $z$ is near to $x_a$ one cannot infer anything about the RCC8 base relation holding between them.

# 5 Nearness and Proximity Spaces

As the relation $\mathsf{NR}$ is intended to model qualitative nearness relations we have to compare them with other formal models of qualitative nearness. A prominent qualitative nearness relation results from the neighborhood concept of topological spaces. A more fine-grained

[1] The authors of [8] state that $\mathsf{NR}$ is at least a weakly symmetrical relation meaning that the symmetry condition holds only if $z$ and $x_a$ are regions of the same partition. But the example above (Fig. 2) shows that the $\mathsf{NR}(x_a,y_a)$ entailing $\mathsf{NR}(y_a,x_a)$ does not hold even for regions $x_a, y_a$ of the same partition.

mathematical approach to nearness is provided by proximity spaces. These date back to ideas of Riesz presented in a congress talk in 1908 [14] and were rediscovered in the fifties by the mathematician Efremovič [5, 6]. He gave the axiomatic definition of a proximity space to become the basis for all following work on proximity spaces. We will not delve into the further development of research on proximity spaces but note that proximity spaces also became an important topic in the area of qualitative spatial reasoning [16, 2, 3, 4]. For a historical overview on proximity spaces (until 1970) the reader may have a look at the introductory chapter of the classic monograph by Naimpally and Warrack [9].

In the following, we will not give the definition of proximity spaces according to Efremovič (see [9, p.7–8]) but rather use the weaker notion of a minimal proximity relation given in [4]. The reason is that the nearness relation considered in this paper is inherently not symmetrical and the total order of partitions is finite, hence induces a discrete approach to nearness which is in the same spirit as the approach of [4].

**Definition 1.** A minimal proximity space $(X, \delta)$ *[4, p. 7] is a structure with a binary relation* $\delta$ *over a set* $X$ *such that the following conditions are fulfilled:*

1. *For all* $A, B \subseteq X$*: If* $A \,\delta\, B$*, then* $A$ *and* $B$ *are nonempty.*
   *(That means, only for non-empty regions does proximity hold).*

2. *For all* $A, B, C \subseteq X$*:*

   (a) $A \,\delta (B \cup C)$ *iff* $A \,\delta\, B$ *or* $A \,\delta\, C$ *(right distribution);*
   *(A is near a union of regions iff it is near one of the regions.)*

   (b) $(A \cup B) \,\delta\, C$ *iff* $A \,\delta\, C$ *or* $B \,\delta\, C$ *(left distribution).*
   *(A union of regions is near a region* $C$ *iff one of the regions is near* $C$*.)*

Proximity spaces are structures that have strong connections to topological spaces. In fact, for a proximity space $(X, \delta)$ a canonical topological space $(X, \tau(\delta))$ can be defined by

$$\tau(\delta) = \{A \subseteq X \mid A \text{ is closed according to (32)}\} \tag{31}$$

and

$$A \subseteq X \text{ is closed under } \delta \text{ iff for all } x \in X\text{: If } x \,\delta\, A, \text{ then } x \in A. \tag{32}$$

Indeed, $(X, \tau(\delta))$ is a topology in the sense that the following conditions are fulfilled: $\{X, \emptyset\} \subseteq \tau(\delta)$; if $A, B \in \tau(\delta)$, then $A \cup B \in \tau$; if $(A_i)_{i \in I}$ is a (possibly infinite) family of sets in $\tau(\delta)$, $A_i \in \tau(\delta)$, then $\bigcap_{i \in I} A_i \in \tau(\delta)$. But, as said before, proximity spaces are finer structures than topological spaces in so far as two different proximities $\delta_1, \delta_2$ may induce the same topology $\tau(\delta_1) = \tau(\delta_2)$.

We will investigate the question whether the nearness operator $\mathsf{NR}$ can be considered as a proximity relation. Clearly, the first condition holds trivially for $\mathsf{NR}$ as we excluded the empty set as a region. The other conditions cannot be applied to $\mathsf{NR}$ directly because the ld regions are not closed with respect to unions. Nonetheless, the following special case of condition ($2a$), in which we consider unions of regions of a partition level whose union makes up a region of the next partition level, may hold. Let be given an ld partition $(a) = (a_i)_{i \in I}$. So all $B \in (a)^{\uparrow}$ can be represented as a union $B = b_1 \cup \cdots \cup b_n$ with $b_j \in (a)$ for $j \in \{1, \ldots, n\}$. Now we may ask whether $\mathsf{NR}(A, b_1)$ or $\ldots$ or $\mathsf{NR}(A, b_n)$ iff $\mathsf{NR}(A, B)$. Of this equivalence only the left-to-right direction holds, as shown by the following proposition.

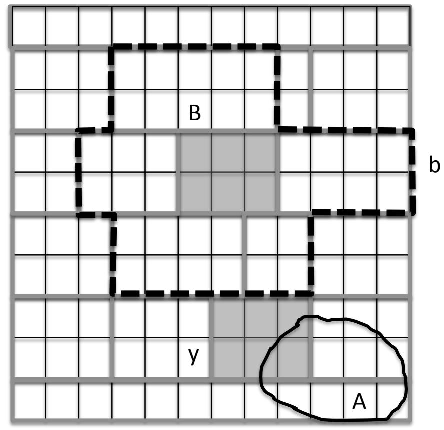


Figure 3: Counterexample for left-to-right direction in proximity condition 2(a)

**Proposition 6.** *Let $(a) = (a_i)_{i \in I}$ be a partition with ld regions. For all regions $A \subseteq X$ and all $B \in (a)^{\uparrow}$ with $B = b_1 \cup \cdots \cup b_n$ for $b_j \in (a)$ and $j \in \{1, \ldots, n\}$ the following entailment holds: if $\mathcal{KB} \models \mathsf{NR}(A, b_1)$ or ... or $\mathcal{KB} \models \mathsf{NR}(A, b_n)$, then $\mathcal{KB} \models \mathsf{NR}(A, B)$.*

*Proof.* Assume that $\mathsf{NR}(A, b_1)$ or ... or $\mathsf{NR}(A, b_n)$ holds. We have to show that $\mathsf{NR}(A, B)$ holds. W.l.o.g assume that $\mathsf{NR}(A, b_1)$ (otherwise rename $b_i$). That means that there is a $y \in (a)$ such that $\mathsf{NR}_{\mathsf{ap}}(y, b_1)$ and $A\{\mathsf{P}, \mathsf{PO}\}y$. Using the definition of $\mathsf{NR}_{\mathsf{ap}}$ this means that there is a $y \in (a)$ and a $b \in (a)^{\uparrow}$ such that $b_1\{\mathsf{P}\}b$ and $y\{\mathsf{P}, \mathsf{EC}\}b$ and $A\{\mathsf{P}, \mathsf{PO}\}y$. But as $b_1$ is already contained in $B$, it follows that $b = B$. Now, for $y$ there is a $D \in (a)^{\uparrow}$ such that $y\{\mathsf{P}\}D$. We show that $A\{\mathsf{P}, \mathsf{PO}\}D$ and $\mathsf{NR}_{\mathsf{ap}}(D, B)$, which immediately implies $\mathsf{NR}(A, B)$. That $y\{\mathsf{P}\}D$ means $y\{\mathsf{EQ}, \mathsf{TPP}, \mathsf{NTPP}\}D$. That $A\{\mathsf{P}, \mathsf{PO}\}y$ means $A\{\mathsf{EQ}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{PO}\}y$. Using the composition table for RCC8 the following relation between $A$ and $D$ follows: $A\{\mathsf{EQ}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{PO}\}D$ which is $A\{\mathsf{P}, \mathsf{PO}\}D$. We have to show $\mathsf{NR}_{\mathsf{ap}}(D, B)$ to complete the proof. As $y\{\mathsf{P}, \mathsf{EC}\}b$, this means $y\{\mathsf{EQ}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{EC}\}b$. If $y\{\mathsf{EQ}, \mathsf{TPP}, \mathsf{NTPP}\}b$, which means that $y\{\mathsf{EQ}, \mathsf{TPP}, \mathsf{NTPP}\}B$, then also $b = B = D$ and so by reflexivity $\mathsf{NR}_{\mathsf{ap}}(D, B)$. If $y\{\mathsf{EC}\}b$ this means $y\{\mathsf{EC}\}B$ or equivalently $B\{\mathsf{EC}\}y$. As $y\{\mathsf{EQ}, \mathsf{TPP}, \mathsf{NTPP}\}D$ it follows using the composition table that $B\{\mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}\}D$. But we know, as $B, D$ are regions of the same partition, that also $B\{\mathsf{EC}, \mathsf{EQ}, \mathsf{DC}\}D$. The intersection gives $B\{\mathsf{EC}\}D$, and hence by Prop. 1 $\mathsf{NR}_{\mathsf{ap}}(B, D)$ follows. □

A simple example (Fig. 3) shows that the other direction of the condition in Prop. 6 does not hold. In Fig. 3 the smallest rectangles represent the finest (ld) partition. The regions $B$ and $y$ are regions of the next upper partition whose regions are represented with grey border lined rectangles. The region $b$ (dotted border line) is the only region of the partition above $B$ (and $y$) that is represented in the figure, and $A$ is an arbitrary region not aligned with the partitions. As one can see, $A$ is near $B$, i.e., $\mathsf{NR}(A, B)$, but $A$ is not near any of the six cells that make up $B$. The reason is that $B$ gives a coarser scaling for nearness than all of its parts $b_i$. Hence, while something may be near w.r.t. a coarser scaling, if it is near with respect to a finer scaling, the converse does not hold.

If we look at condition (2b), then there is a chance that both directions may hold, because on both sides of the biimplication the same $C$ occurs as the second (level determining) position of $\mathsf{NR}$. But here again, we can show only one direction, the direction which is opposite to the previous one.

**Proposition 7.** *For all $A, B, C \subseteq X$: If $\mathcal{KB} \models \mathsf{NR}(A \cup B, C)$, then $\mathcal{KB} \models \mathsf{NR}(A, C)$ or $\mathcal{KB} \models \mathsf{NR}(B, C)$.*

*Proof.* Assume $\mathcal{KB} \models \mathsf{NR}(A \cup B, C)$, i.e., $(A \cup B)\{\mathsf{P}, \mathsf{PO}\}y$ and $\mathsf{NR}_{\mathsf{ap}}(y, C)$ for an ld region $y$ in the same partition as $C$. But then either $A\{\mathsf{P}, \mathsf{PO}\}y$ or $B\{\mathsf{P}, \mathsf{PO}\}y$, which follows from the definition of $A \cup B$ as sum of $A$ and $B$. Lets see why this is the case. If $(A \cup B)\{\mathsf{P}\}C$, then even $A\{\mathsf{P}\}C$ and $B\{\mathsf{P}\}C$ holds. So assume $(A \cup B)\{\mathsf{PO}\}C$. We have to show $A\{\mathsf{P}, \mathsf{PO}\}C$ or $B\{\mathsf{P}, \mathsf{PO}\}C$. So for reductio ad absurdum assume that not $A\{\mathsf{P}, \mathsf{PO}\}C$ and not $B\{\mathsf{P}, \mathsf{PO}\}C$, which means that $A\{\mathsf{TPPi}, \mathsf{NTPPi}, \mathsf{DC}, \mathsf{EC}\}C$ and $B\{\mathsf{TPPi}, \mathsf{NTPPi}, \mathsf{DC}, \mathsf{EC}\}C$. We can exclude the cases for $\mathsf{TPPi}$, $\mathsf{NTPPi}$ as this would immediately imply $C\{P\}(A \cup B)$—contradicting $A \cup B\{\mathsf{PO}\}C$. So there are four cases which all lead to contradictions.

1. $A\{\mathsf{DC}\}C$ and $B\{\mathsf{DC}\}C$: Because $(A \cup B)\{\mathsf{PO}\}C$ there is a $w$ such that $w\{\mathsf{P}\}(A \cup B)$ and $w\{\mathsf{P}\}C$. But the first implies $w\{C\}A \cup B$ hence $w\{C\}A$ or $w\{C\}B$ and hence not $A\{\mathsf{DC}\}C$ or not $B\{\mathsf{DC}\}C$, contradiction.

2. $A\{\mathsf{DC}\}C$ and $B\{\mathsf{EC}\}C$: Because $(A \cup B)\{\mathsf{PO}\}C$ there is a $w$ such that $w\{\mathsf{P}\}(A \cup B)$ and $w\{\mathsf{P}\}C$. Clearly $A\{\mathsf{DC}\}w$. Now we consider all possible base relations between $B$ and $w$. $B\{\mathsf{TPP}, \mathsf{NTPP}\}w$ cannot be the case because then $B\{\mathsf{P}\}C$ would follow. $B\{\mathsf{DC}\}w$ cannot be the case either because $w\{\mathsf{P}\}(A \cup B)$, so especially $w\{\mathsf{C}\}(A \cup B)$, hence $w\{\mathsf{C}\}A$ or $w\{\mathsf{C}\}B$. But as $A\{\mathsf{DC}\}w$ it follows that $w\{\mathsf{C}\}B$, contradiction. Now assume $B\{\mathsf{EC}\}w$. Because of the non-atomicity axiom there is a $w'$ such that $w'\{\mathsf{NTPP}\}w$, and it fulfills $w'\{\mathsf{P}\}(A \cup B)$ and $w'\{\mathsf{DC}\}A$ and $B\{\mathsf{DC}\}w'$. But this leads to the same contradiction as for $B\{\mathsf{DC}\}w$. We move on and assume $B\{\mathsf{PO}\}w$. So there is a $w''$ such that $w''\{\mathsf{P}\}B$ and $w''\{\mathsf{P}\}w\{\mathsf{P}\}C$. But this contradicts $B\{\mathsf{EC}\}C$. So for $B$ only $B\{\mathsf{EQ}, \mathsf{TPPi}, \mathsf{NTPPi}\}w$ is left, but this contradicts $B\{\mathsf{EC}\}C$.

3. $A\{\mathsf{EC}\}C$ and $B\{\mathsf{DC}\}C$: Symmetrical argumentation to the case before.

4. $A\{\mathsf{EC}\}C$ and $B\{\mathsf{EC}\}C$: Again because $(A \cup B)\{\mathsf{PO}\}C$, there is a $w$ such that $w\{\mathsf{P}\}(A \cup B)$ and $w\{\mathsf{P}\}C$. Now we go through all possible base relations between $A$ and $w$. Case $A\{\mathsf{DC}\}w$: This leads to the contradictions as derived in case (2). Case $A\{\mathsf{EC}\}w$: Choose $w'$ such that $w'\{\mathsf{NTPP}\}w$. Then $A\{\mathsf{DC}\}w'$ and the same arguments work as before. Now consider the case $A\{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}, \mathsf{TPPi}, \mathsf{NTPPi}\}w$: This cannot hold as it would directly contradict $A\{\mathsf{EC}\}C$. Case $A\{\mathsf{PO}\}w$: This contradicts $A\{\mathsf{EC}\}C$ too, because then there is a $w''\{\mathsf{P}\}w\{\mathsf{P}\}C$ and $w''\{\mathsf{P}\}A$ leading to $A\{\mathsf{EQ}, \mathsf{PO}\}C$.

□

The proposition above justifies the definition and investigation of structures which we call *weak right-scaled proximity spaces*. So Propositions 6 and 7 say that $(X, \mathsf{NR})$ (for $X$ being a region in BRCC) is (almost) a weak proximity space.

**Definition 2.** *The structure $(X, \delta)$ is a* weak right-scaled proximity space *iff $\delta$ is a binary relation over $X$ such that the following conditions are fulfilled.*

1. *For all $A, B \subseteq X$: If $A \, \delta \, B$, then $A$ and $B$ are nonempty.*

2. *For all $A, B, C \subseteq X$:*

   (a) *If $A\,\delta\,B$ or $A\,\delta\,C$, then $A\,\delta(B \cup C)$;*

   (b) *if $(A \cup B)\,\delta\,C$, then $A\,\delta\,C$ or $B\,\delta\,C$.*

Similarly one can define the dual notion of left scaled proximity spaces by switching the directions in the condition (2a) and (2b).

The other direction of the implication in Prop. 7 does not hold, because $A \cup B$ may become too big. Take for example an $A$ such that $\mathsf{NR}(A, C)$, and assume $C$ is not $X$; hence there is a $y$ such that $A\{\mathsf{P}, \mathsf{PO}\}y$ and $\mathsf{NR}_{\mathsf{ap}}(y, C)$ and let $B = X$. Then $A \cup B = X$ and for all ld regions $y'$ other than $X$ it holds that $y'\{\mathsf{TPP}, \mathsf{NTPP}\}X$. So in particular, there is no $y'$ for which $\mathsf{NR}_{\mathsf{ap}}(y', C)$ and $A \cup B = X\{\mathsf{P}, \mathsf{PO}\}y'$. Even if one restricts the left argument to unions of ld regions that make up an ld partition of the next higher level, a counterexample can be constructed.

# 6 Extensions and Modifications of the Nearness Relation

An unwanted feature of the nearness rule (23) is that it allows only ld regions as second arguments. Therefore, we define a new nearness relation $\widetilde{\mathsf{NR}}$ that allows for arbitrary regions in both argument positions—though still the second argument will determine the scaling context for nearness. Region $z$ is considered to be near region $x$ iff $z$ is $\mathsf{NR}$-near the ld region of smallest level containing $x$, formally:

$$\widetilde{\mathsf{NR}}(z, x) \text{ iff } \mathsf{NR}(z, \tilde{x}) \text{ where } \tilde{x} \text{ is the } \mathsf{P}\text{-smallest ld region s.t. } x\{\mathsf{P}\}\tilde{x}. \tag{33}$$

We say that a region $x$ is of level $j$ iff $\tilde{x}$ is an ld region of the partition level $j$. As the following proposition shows, the shift from $\mathsf{NR}$ to the extended $\widetilde{\mathsf{NR}}$ is conservative in the sense that the properties of $\mathsf{NR}$ are preserved by $\widetilde{\mathsf{NR}}$.

**Proposition 8.** *The extended nearness relation $\widetilde{\mathsf{NR}}$ has the following properties:*

1. *For all ld regions $y_a, x_a$: If $\mathcal{KB} \models \mathsf{NR}_{\mathsf{ap}}(y_a, x_a)$, then $\mathcal{KB} \models \widetilde{\mathsf{NR}}(y_a, x_a)$.*

2. *For all $z$ and ld $x_a$: If $\mathcal{KB} \models \mathsf{NR}(z, x_a)$, then $\mathcal{KB} \models \widetilde{\mathsf{NR}}(z, x_a)$.*

3. *$\widetilde{\mathsf{NR}}$ is reflexive: Fo all $z$: $\mathcal{KB} \models \widetilde{\mathsf{NR}}(z, z)$.*

4. *For all $A, B, C \subseteq X$: If $\mathcal{KB} \models \widetilde{\mathsf{NR}}(A \cup B, C)$, then $\mathcal{KB} \models \widetilde{\mathsf{NR}}(A, C)$ or $\mathcal{KB} \models \widetilde{\mathsf{NR}}(B, C)$.*

5. *For all $A, B, C \subseteq X$: If $\mathcal{KB} \models \widetilde{\mathsf{NR}}(A, B)$ or $\mathcal{KB} \models \widetilde{\mathsf{NR}}(A, C)$, then $\mathcal{KB} \models \widetilde{\mathsf{NR}}(A, B \cup C)$.*

*Proof.* The proofs of the first three items are easy. The proof of the fourth item follows directly from Prop. 7 and the definition of $\widetilde{\mathsf{NR}}$. For the proof of the last item note that if $\mathcal{KB} \models \widetilde{\mathsf{NR}}(A, B)$ or $\mathcal{KB} \models \widetilde{\mathsf{NR}}(A, C)$, then $\mathcal{KB} \models \mathsf{NR}(A, \tilde{B})$ or $\mathcal{KB} \models \mathsf{NR}(A, \tilde{C})$. W.l.o.g. assume that the first disjunct holds. Then there is a region $y$ of the same partition as $\tilde{B}$ such that $\mathsf{NR}_{\mathsf{ap}}(y, \tilde{B})$ and $A\{\mathsf{P}, \mathsf{PO}\}y$. That means there is a region $b$ in the next level above $y$ and $\tilde{B}$ such that $\tilde{B}\{\mathsf{P}\}b$ and $y\{\mathsf{P}, \mathsf{EC}\}b$. Now $b\{\mathsf{P}\}\widetilde{B \cup C}$. If $b\{\mathsf{EQ}\}\widetilde{B \cup C}$, then let $y'$ be the region of the next upper level in which $y$ is contained. Then $y'\{\mathsf{EQ}, \mathsf{EC}\}\widetilde{B \cup C}$ (so $\mathsf{NR}_{\mathsf{ap}}(y', \widetilde{B \cup C})$) and $A\{\mathsf{P}, \mathsf{PO}\}y'$, hence $\mathsf{NR}(A, \widetilde{B \cup C})$ and so $\widetilde{\mathsf{NR}}(A, B \cup C)$. If $b\{\mathsf{TPP}, \mathsf{NTPP}\}\widetilde{B \cup C}$, then $y\{\mathsf{P}\}\widetilde{B \cup C}$ and hence $A\{\mathsf{P}, \mathsf{PO}\}\widetilde{B \cup C}$. As $\mathsf{NR}_{\mathsf{ap}}(\widetilde{B \cup C}, \widetilde{B \cup C})$ it follows again $\mathsf{NR}(A, \widetilde{B \cup C})$ and hence $\widetilde{\mathsf{NR}}(A, B \cup C)$. □

We will now look at a further modification of the nearness relation that is guided by additional axioms investigated in the context of proximity relations. The following axiom of an (Efremovič) proximity relation $\delta$ (see [9, p.7–8]) asserts that a nonempty intersection of sets is sufficient for them to count as near.

$$\text{If } A \cap B \neq \emptyset, \text{ then } A\,\delta\,B \text{ (and } B\,\delta\,A). \tag{34}$$

As we have seen above, this property does not hold for $\mathsf{NR}$ (and so not for $\widetilde{\mathsf{NR}}$), as there may be regularly closed regions $A$ and $B$ such that $A\{\mathsf{EC}\}B$ but not $\mathsf{NR}(A, B)$. But at least we can show the following weaker entailment:

**Proposition 9.** *For regions $A, B$ which touch each other ($A\{\mathsf{EC}\}B$) at least one of* $\widetilde{\mathsf{NR}}(A, B)$ *or* $\widetilde{\mathsf{NR}}(B, A)$ *holds.*

*Proof.* We consider the case whether $B$ is an ld region (i.e., $\tilde{B} = B$) or not (i.e., $\tilde{B} \neq B$)—starting with the easier second case. So $B \neq \tilde{B}$ and $A\{\mathsf{EC}\}B$ implies that $A\{\mathsf{P}, \mathsf{PO}\}\tilde{B}$. As $\mathsf{NR}_{\mathsf{ap}}(\tilde{B}, \tilde{B})$ it follows that $\mathsf{NR}(A, \tilde{B})$, hence $\widetilde{\mathsf{NR}}(A, B)$.

In the other case $B = \tilde{B}$ and $A\{\mathsf{EC}\}B$ implies that $A\{\mathsf{EQ}, \mathsf{TPP}\}(X \setminus B)$. We first consider the case that $A\{\mathsf{TPP}\}(X \setminus B)$. Then there is a $B' \subseteq X \setminus B$ which is in the same partition level as $B$ and $A\{\mathsf{P}, \mathsf{PO}\}B'$ and $B'\{\mathsf{EC}\}B$ (especially $\mathsf{NR}_{\mathsf{ap}}(B', B)$). Hence $\mathsf{NR}(A, B)$ and $\widetilde{\mathsf{NR}}(A, B)$. Now, we consider the case $A\{\mathsf{EQ}\}(X \setminus B)$, i.e., $A = X \setminus B$. If $X \setminus B$ is an ld region on the same level as $B$, then already $\mathsf{NR}_{\mathsf{ap}}(A, B)$ and hence $\mathsf{NR}(A, B)$ and $\widetilde{\mathsf{NR}}(A, B)$. If If $X \setminus B$ is not an ld region, then it is a union of regions $b_i$ on the same level as $B$. But then $\tilde{A} = X$ and hence $\mathsf{NR}(B, \tilde{A})$ and so $\widetilde{\mathsf{NR}}(B, A)$. □

But clearly we can define a new nearness relation $\widehat{\mathsf{NR}}$ that extends $\mathsf{NR}$ and fulfills the axiom in (**??**) in the following way:

$$\widehat{\mathsf{NR}}(z, x) \text{ iff either } \mathsf{C}(z, x) \text{ } (z \text{ and } x \text{ are connected) or } \widetilde{\mathsf{NR}}(z, x). \tag{35}$$

Clearly $\widetilde{\mathsf{NR}} \subseteq \widehat{\mathsf{NR}}$, $\widehat{\mathsf{NR}}$ fulfills the separation axiom and one can easily show that all properties of $\widetilde{\mathsf{NR}}$ mentioned in Prop. 8 also hold for $\widehat{\mathsf{NR}}$.[2] Moreover, this nearness relation fulfills even the equivalence in condition (2b) of the definition for proximity spaces. Hence, we have a model of a structure we want to call a *right-scaled proximity space.*

**Definition 3.** *A* right-scaled proximity space $(X, \delta)$ *is a structure with a set $X$ and a binary relation $\delta$ over $X$ such that $\delta$ fulfills the following conditions:*

[2]Another nearness notion which also fulfills the condition that "$A \cap B \neq \emptyset$ entails $A\delta B$" could start with a redefinition of the a-priori relation: $\mathsf{NR}^{\mathsf{new}}_{\mathsf{ap}}(y_a, x_a)$ iff there is $b \in (a)^{\uparrow}$ such that $y_a\{\mathsf{P}\}b$ and $x_a\{\mathsf{P}\}b$. Then one defines $\mathsf{NR}^{\mathsf{new}}(z, x)$ iff there is $y$ in the level of $\tilde{x}$ such that $\mathsf{NR}^{\mathsf{new}}_{\mathsf{ap}}(y, \tilde{x})$ and $\mathsf{C}(z, y)$.

1. *For all* $A, B \subseteq X$*: If* $A \,\delta\, B$*, then* $A$ *and* $B$ *are nonempty.*

2. *For all* $A, B, C \subseteq X$*:*

   (a) *If* $A \,\delta\, B$ *or* $A \,\delta\, C$*, then* $A \,\delta (B \cup C)$*;*

   (b) $(A \cup B) \,\delta\, C$ *iff* $A \,\delta\, C$ *or* $B \,\delta\, C$*.*

3. *For all* $A, B \subseteq X$*: If* $A \cap B \neq \emptyset$*, then* $A \,\delta\, B$ *(and* $B \,\delta\, A$*).*

Now the main facts concerning $\widehat{\mathsf{NR}}$ can be restated by saying that it is a right-scaled proximity relation.

**Proposition 10.** *Let* $X$ *be a region (in BRCC). Then* $(X, \widehat{\mathsf{NR}})$ *is a right-scaled proximity space.*

The conditions stated in a right-scaled proximity space are not strong enough to define a canonical topological space as is done for proximity spaces (see above). But nonetheless the nearness relation can be seen as an interleaving of level-fixed nearness relations. This will be explicated in the following. Let be given a total ordering of partitions $(a_i^j)_{i \in I_j}$ over $X$, $1 \leq j \leq n$. For every partition level $j$ we will define nearness relations $\widehat{\mathsf{NR}}^j$ between arbitrary regions $z_1, z_2 \subseteq X$.

$$\widehat{\mathsf{NR}}^j(z_1, z_2) \text{ iff there is a } y \text{ of level } j \text{ s.t.} \, \widehat{\mathsf{NR}}(z_1, y) \text{ and } \widehat{\mathsf{NR}}(z_2, y). \tag{36}$$

These nearness relations are symmetric and are ordered with respect to inclusion and fulfill the conditions of a minimal proximity space. Their definition is similar to the composition $R \circ R^{-1}$ of a binary relation $R$ with its inverse.

**Proposition 11.** *The level-fixed nearness relations* $\mathsf{NR}^j$ *fulfill the following conditions:*

1. *Every* $\widehat{\mathsf{NR}}^j$ *is a (symmetric) proximity relation (and hence induces a topology).*

2. *If* $i \leq j$*, then* $\widehat{\mathsf{NR}}^i \subseteq \widehat{\mathsf{NR}}^j$*.*

3. *If* $\widehat{\mathsf{NR}}(z_1, z_2)$ *and* $z_2$ *is of level* $j$*, then* $\widehat{\mathsf{NR}}^j(z_1, z_2)$*.*

*Proof.* Ad 1: Symmetry follows from the symmetry of "and". The fulfillment of condition 1 for proximity spaces is clear. In order to show the fulfillment of the conditions (2a) and (2b) it suffices to show one of them (we show (2b)) because of the symmetry of $\widehat{\mathsf{NR}}^j$. So $\widehat{\mathsf{NR}}^j(A \cup B, C)$ iff there is a $y$ of level $j$ such that $\widehat{\mathsf{NR}}(A \cup B, y)$ and $\widehat{\mathsf{NR}}(C, y)$. This is equivalent to saying there are $y_1$ and $y_2$ of level $j$ such that $(A \cup B)\{\mathsf{C}\}y_1$ and $\mathsf{NR}_{\mathsf{ap}}(y_1, y)$ as well as $C\{\mathsf{C}\}y_2$ and $\mathsf{NR}_{\mathsf{ap}}(y_2, y)$. But per definitionem $(A \cup B)\{\mathsf{C}\}y_1$ iff $A\{\mathsf{C}\}y_1$ or $B\{\mathsf{C}\}y_1$, so that the equivalence with $(\widehat{\mathsf{NR}}^j(A, C)$ or $\widehat{\mathsf{NR}}^j(B, C))$ follows.

Ad 2: Let $\widehat{\mathsf{NR}}^i(z_1, z_2)$, i.e., there is a $y$ of level $i$ s.t. $\widehat{\mathsf{NR}}(z_1, y)$ and $\widehat{\mathsf{NR}}(z_2, y)$. There is a $y'$ of level $j$ such that $y\{\mathsf{P}\}y'$. As $\widehat{\mathsf{NR}}$ is a right-scaled proximity relation, it follows that $\widehat{\mathsf{NR}}(z_1, y')$ and $\widehat{\mathsf{NR}}(z_2, y')$ which shows $\widehat{\mathsf{NR}}^j(z_1, z_2)$.

Ad 3: Let $\widehat{\mathsf{NR}}(z_1, z_2)$ and let $z_2$ be of level $j$. Then for $y = z_2$ we have $\widehat{\mathsf{NR}}(z_1, y)$ and (because of reflexivity) $\widehat{\mathsf{NR}}(z_2, y)$ showing $\widehat{\mathsf{NR}}^j(z_1, z_2)$. □

As a résumé we may state that though the nearness relation $\widehat{\mathsf{NR}}$ is not a (minimal) proximity relation each of its levels induces a proximity relation $\widehat{\mathsf{NR}}^j$ extending $\widehat{\mathsf{NR}}$.

# 7 Conclusion and Future Work

The analysis of the nearness relations $\mathsf{NR}$ and $\mathsf{NR}_{\mathsf{ap}}$ which are simpler versions of the nearness relations of [8, 7] has revealed some properties which they share with natural nearness concepts but also some properties which distinguish them from nearness relations formalized by proximity spaces. The limited applicability of $\mathsf{NR}$ to ld regions could be overcome by extending it to the relation $\widetilde{\mathsf{NR}}$. The relation $\widetilde{\mathsf{NR}}$ could be shown to fulfill some subset of the axioms for minimal proximity spaces which resulted in the definition of structures we termed weak right-scaled proximity spaces. Lessening the difference to proximity relations even further, we defined a new relation $\widehat{\mathsf{NR}}$ which fulfills the axioms of what we have termed a right-scaled proximity space. The relation $\widehat{\mathsf{NR}}$ is a good candidate as an element of a controlled natural language interface to geographical data because it provides a good approximation of the nearness as modelled by proximity spaces. But its exact corresponding axiomatic specification is given by right-scaled proximity spaces.

The (weak) right-scaled structures have still to be investigated mathematically. We have discussed the proximity spaces independently of the RCC background theory. An equivalent representation of RCC by boolean contact algebras [15] provides the basis for the investigation of structures that are boolean contact algebras equipped with proximity relations. Motivated by our nearness relations we plan to investigate combinations of boolean contact algebras with right-scaled proximity relations. These structures will provide the logical framework in which one can properly formulate and answer the question whether the nearness relation $\widehat{\mathsf{NR}}$ is a canonical model for right-scaled proximity relations. That is, if there is a relation $\delta$ in a boolean contact algebra that fulfills the axioms of right-scaled proximity spaces, can it be represented equivalently as the relation $\widehat{\mathsf{NR}}$? A positive answer would show that the nearness relation $\widehat{\mathsf{NR}}$ is a prominent member of right-scaled proximity relations.

Further future work concerns the reincorporation of functional regions as used in the original nearness definitions of [8, 7]. These more complex relations share the scaling dependence with the relation $\widehat{\mathsf{NR}}$ but may show different behavior depending on how the functional regions are embedded within the ld regions. This latter point leads to another task that deals with the robustness of the nearness relations; one should investigate the effects a change of the partitions (or the functional regions) has on the nearness relation: in particular one should investigate conditions under which two total orders of partitions induce the same nearness relation.

Another starting point for future work is the observation that original rules for nearness relations only give sufficient conditions. Though there may be no conditions that are both sufficient and necessary for nearness, one may investigate additional rules that present sufficient conditions for the negation of nearness (which one could equivalently describe as apartness). These rules can be thought of as negative integrity constraints as generally $A_1 \wedge \cdots \wedge A_n \rightarrow \neg B$ is equivalent to $A_1 \wedge \cdots \wedge A_n \wedge B \rightarrow \bot$. In this framework, it may be the case that there are still regions $x, y$ for which the knowledge base neither implies $\mathsf{NR}(x, y)$ nor implies $\neg\, \mathsf{NR}(x, y)$. The nearness relation $\mathsf{NR}$ thus becomes a partial relation ([8] speak of “vague” relation). In this framework one can define weaker notions of symmetry and transitivity in the same spirit as the approach of Worboys [17]. Worboys uses the four-valued semantics of [1], which is based on the truth values $T$ (true), $F$ (false), $N$ (neither true nor false), $B$ (both true and false), in order to define weak symmetry as follows: If the value of $\mathsf{NR}(x, y)$ is $T$, then the value of $\mathsf{NR}(y, x)$ is not $F$ (but may be one of $T, N, B$). We could mimic these definitions in a three-valued semantics corresponding to the three possibilities of a fact $A$ being entailed by $\mathcal{KB}$ or being falsified ($\neg A$ is entailed by $\mathcal{KB}$) or neither of the two former cases holding. Now one could, e.g., test whether $\mathsf{NR}$ is weakly symmetrical in the following sense: For all $x, y$: If $\mathcal{KB} \models \mathsf{NR}(x, y)$, then not $\mathcal{KB} \models \neg\,\mathsf{NR}(y, x)$, i.e., for $x, y$ the knowledge base $\mathcal{KB}$ may not entail $\mathsf{NR}(x, y)$, but $\mathcal{KB}$ is not strong enough to prove the negation $\neg\,\mathsf{NR}(x, y)$.

A last future work package concerns the implementation aspect and the computational feasibility aspect of answering queries that contain nearness relations. If the nearness relation is given only by rules stating sufficient conditions and there is no other background terminology in which the nearness relation may be used as well, then query answering is reducible to a macro expansion of the relations (see the implemented algorithm in [7]). Otherwise query answering may become quite harder and one has to deal with the question whether the queries can be rewritten to semantically equivalent first order logic queries—in the same spirit as that of [11].

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