Dynamics of a Nearness Relation—First Results

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Abstract. The system of administrative units for a state like Switzerland can be formally described by a totally ordered set of nested partitions where, e.g., the municipalities make up a finer partition than the partition induced by districts. Based on these partitions one can define binary non-symmetric nearness relations between regions in which the second argument determines the granularity (or the scaling level) w.r.t. which the first-argument region is to be considered near or not. The logical properties of such nearness relations, especially w.r.t. their relation to proximity relations, have been worked out in the authors' contribution to ECAI 2012. Referring to these properties, in this paper we extend the investigation to the dynamics of the nearness relation. In particular, we investigate how a change within the total order of partitions (e.g., two municipalities are merged) affects the induced nearness relation.

1 INTRODUCTION

The nearness relation whose dynamics we are going to discuss is defined on the basis of a hierarchy of nested regions which make up a total order of partitions [3], [7], [8]. Typical examples of such total orders of nested partitions are made up of administrative units where the administrative units in a rougher granularity are the sums (unions) of administrative units of the lower level. As an example think of two partitions of Switzerland, where the first partition consists of municipalities and where the second consists of districts. All districts are municipalities or are unions of two or more municipalities.

Every partition provides a granularity or scale w.r.t. which the nearness of two regions are declared; the main idea is to consider one of the arguments (we took the second one) to determine the scaling context that is the level on the ground of which two regions are defined to be near or not. There are different ways to exploit the nested partitions pc (which mathematically is a totally ordered set of partitions and hence termed *partition chain*) for defining nearness relations. We will fix a specific type of nearness relation NR_{pc} induced by a partition chain pc which has some desirable properties.

Having constructed such a nearness relation one can consider its properties in a mathematically abstract way by declaratively specifying properties of a binary relation δ in a formal language like first order logic. In previous work, we described the properties that every right-scaled proximity nearness relation NR_{pc} induced by a partition chain pc has [7], [8]. In this paper, we add some further properties of this type and extend the investigations in two ways: we describe the local dynamics of nearness relations, that is we describe how a change from one region to another (in the second argument) affects the set of regions considered to be near. For this purpose, we describe properties that directly refer to the given (unchanged) partition chain. Second we investigate the question how the change of the partition chain affects the nearness relation, i.e., we investigate the global dynamics of nearness. More concretely, assume a partition chain pc_1 is changed to a new partition chain pc_2; what can we say about the change from the induced nearness relation NR_{pc_1} to the induced nearness relation NR_{pc_2}? In particular, one can ask what kind of change transitions $pc_1 \rightsquigarrow pc_2$ do not change the nearness relation, $\mathsf{NR}_{pc_1} = \mathsf{NR}_{pc_2}$, or between what regions (on what level) does a change of the total orderings affect the nearness in between them. Similar problems have been tackled by [4] and especially [11], which considers the global dynamics of tree-like spatial configurations.

The change transition $\rightsquigarrow$ between total orders are not allowed to be arbitrary transitions but some intuitive changes which have corresponding real world counterparts. In particular, the kind of changes that are worth being investigated are the merger of regions, the switch of levels, the additions of partitions etc.

Investigations into this kind of relation are necessary for a formal theory of dynamics of nearness. In particular such a theory provides a formal grounding for optimizations within a cognitive agent that bases its nearness relation on partition chains; rather than recalculating the nearness relation between all regions in case the agent moves around (local change) or a partition chain is updated (global change) it directly uses the knowledge on regions between which the nearness relation is expected to have changed. In this work, we lay the foundations for such a theory; thereby we give preliminary results on the local dynamics and the global dynamics of the nearness relation. Concerning the latter we focus on the effects on the nearness relation resulting from merging two regions in the same partition level.

The paper is structured as follows. In Section 2 we describe the main structure for our nearness relations, the partition chain. A specific nearness relation is defined and illustrated in Sect. 3. The following two sections 4 and 5 describe properties of the nearness relation, the former more abstractly by referring only to properties describable by an abstract binary relation, the latter referring also to the underlying partition chain, thereby providing insights into the local dynamics of the nearness relation. The last section before the conclusion starts the preliminary investigation into dynamic aspects of the nearness relation.

2 NORMAL PARTITION CHAINS

In this section we recapitulate the notion of a partition chain underlying the formal framework of a nearness relation as developed in [8], and specify the special class of normal partition chains, which is the main structure for the nearness relation. Different from [8], in this paper, we abstract from the region connection calculus [9], and hence define partition chains and nearness relations only on the basis

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of the usual set theoretical notions.

As we want to allow more cells to be on different levels, we have to type the sets. This is needed for modelling situations in the real world where the same spatial region may have two different administrative functions.[4] Hence we define the following notion of partition:

Definition 1 (partition). *Under a partition of a set X on level $i \in \mathbb{N}$ we understand a family of pairs $(i, a_j)_{j \in J}$ such that $(a_j)_{j \in J}$ is a (set) partition of X, i.e., $X = \biguplus_{j \in J} a_j$ where $\uplus$ indicates a union of disjoint sets and J is a finite index set. A pair $c = (i, a_j)$ is called a cell of level i. Its level i is denoted $l(c)$ and its underlying set a_j (the second argument) is denoted $us(c)$. The usual mathematical notion of a partition will be called* set partition*.*

Now we look at $n + 1$ different partitions of X that are nested or more formally: totally ordered from 0 to n. This is concretised in the following definition.

Definition 2 (partition chain). *Let be given $n+1$ different partitions of X where all partitions have only finitely many cells. We call this set of partitions a* partition chain *pc iff*

1. *all cells $(i + 1, a_j)$ of level $i + 1$ (for $i \in \{0, \dots, n - 1\}$) are unions of i-level cells, i.e., there exist (i, b_k), $k \in K$, such that $a_j = \biguplus_{k \in K} b_k$;*[5]
2. *and the last partition (level n) is made up by (X).*

According to this assumption, every cell has a unique upper cell. For a cell (i, a_j) (with $1 \leq i \leq n - 1$) let $(i, a_j)^{\uparrow, pc} = (i + 1, a_k)$ be the unique cell of the upper level in this partition chain pc such that $a_j \subseteq a_k$. For the cell of level n set $(n, X)^{\uparrow, pc} = (n, X)$. We call $(i, a_j)^{\uparrow, pc}$ the upper cell *of (i, a_j). If the partition chain is clear from the context, we write $(i, a_j)^{\uparrow}$ for $(i, a_j)^{\uparrow, pc}$.*

Between cells (i, a) and (j, b) (perhaps of different partition chains) we define an order $\leq$ by setting $(i, a) \leq (j, b)$ iff $i \leq j$ and $a \subseteq b$.

This definition is too general in order to be used for an interesting nearness notion as it also allows for a configuration where all underlying sets of cells in a partition re-occur in the partition of the next upper level. An example for such an unusual partition is given as follows: let $X = a_1 \uplus a_2$ and let for $i \in \{0, 1, 2, 3\}$ be given the partition a^i of level i by $((i, a_1), (i, a_2))$; the partition of level 3 shall be $(3, X)$. We exclude such partition chains by defining the notion of a *normal partition chain*, in which it is allowed that a set is the underlying sets of two different levels i, $i + 1$, but only if the underlying set partitions on level i and $i + 1$ are different.

Definition 3 (normal partition chain). *A partition chain is* normal *iff all set partitions underlying the partitions are pairwise distinct.*

In practical real-world applications the induced partition can pretty safely assumed to be normal as otherwise a distinction between the administrative units would not even be introduced. But, as in the case of normal partition chains, it may be the case that the same region has two different administrative functions.

[4] Note that in case of the region connection calculus, this typification can be handled directly by allowing non-strict models [10], that is models in which two objects may stand in EQ-relation (same spatial extension) without being identical.

[5] Perhaps in future work we have to reverse the ordering so that we can also consider infinite ordering of partitions: then we may have at level 0 the roughest partition X and at higher levels more fine-grained partitions ad infinitum.

Due to the fact that the total order is finite and that all partitions are finite one can easily describe all possible normal partition chains. For illustration of the notion of a normal partition chain, we will describe the normal partition chains induced by a given set partition (a) with n cells for different n up to $n = 3$.

Example 1. *In this example we write $i : (a)$ as shorthand for partitions $(i, a_j)_{j \in J}$.*

- *If $n = 1$, then the partition (a) is the partition (X) and we do have only the order of partitions containing (X).*
- *Let $n = 2$, i.e. let $X = a_1 \uplus a_2$. We can only have the order of partitions $0 : (a) \leq 1 : (X)$ and $0 : (X)$.*
- *Let $n = 3$, $X = a_1 \uplus a_2 \uplus a_3$. We may have*
  - $0 : (a) \leq 1 : (X)$
  - $0 : (a_1, a_2 \cup a_3) \leq 1 : (X)$
  - $0 : (a_1 \cup a_2, a_3) \leq 1 : (X)$
  - $0 : (a_1 \cup a_3, a_2) \leq 1 : (X)$
  - $0 : (a) \leq 1 : (a_1, a_2 \cup a_3) \leq 2 : (X)$
  - $0 : (a) \leq 1 : (a_1 \cup a_2, a_3) \leq 2 : (X)$
  - $0 : (a) \leq 1 : (a_1 \cup a_3, a_2) \leq 2 : (X)$

3 NEARNESS BASED ON PARTITION CHAINS

Having defined the main structure for a hierarchical nearness relation, we are now in a position to define the notions of apriori nearness and (general) nearness w.r.t. a partition chain as follows:

Definition 4 ((apriori) nearness NR). *Let be given a partition chain pc. Cell $c_1 = (i, a_1)$ is* apriori near *$c_2 = (i, a_2)$, $\mathsf{NR^{ap}}_{pc}(c_1, c_2)$ for short, iff there is a cell $(i + 1, b)$ of level $i + 1$ such that, $a_1, a_2 \subseteq b$, i.e., iff the upper cells of c_1, c_2 are the same. An arbitrary set a is near a cell $c_1 = (i, a_1)$ of level i, $\mathsf{NR}_{pc}(a, c_1)$ for short, iff there is a cell $c_2 = (i, a_2)$ of the same level of c_1 such that $\mathsf{NR^{ap}}_{pc}(c_1, c_2)$ and $a \cap a_2 \neq \emptyset$.*

For an arbitrary set $b \neq \emptyset$ let $\tilde{b}^{pc}$ denote the cell (i, a_j) such that $b \subseteq a_j$ and i is minimal. The integer $i = l_{pc}(b)$ is called the level of b in pc. For arbitrary sets a, b we define nearness by:

$$\mathsf{NR}_{pc}(a, b) \text{ iff } \mathsf{NR}_{pc}(a, \tilde{b}^{pc})$$

If the partition chain pc is unique in the used context, then we do not mention it in the subscripts.

As a shorthand for $(\tilde{b}^{pc})^{\uparrow, pc}$ we write $b^{\Uparrow, pc}$ or even shorter $b^{\Uparrow}$.

Note that we excluded the empty set as a second argument b, as we cannot define $\tilde{\emptyset}$. For the empty set as left-hand argument we get that not $\mathsf{NR}(\emptyset, b)$ for all $b \subseteq X$.

The apriori nearness relation underlying the nearness relation of [8] is different from the one defined here. We chose to work with this definition as it has a very simple equivalent form which does not need the detour with apriori nearness. An arbitrary set a is near a cell (i, a_1) if the intersection with the underlying set of the upper cell of (i, a_1) is non-empty.

Proposition 1. *The nearness relation NR can be equivalently described as follows:*

$$\mathsf{NR}(a, b) \text{ iff } a \cap us(b^{\Uparrow}) \neq \emptyset \tag{1}$$

Proof. If $a \cap us(\tilde{b}^{\uparrow}) \neq \emptyset$, then there is a cell $c_1 = (i, b_1)$ in the same level as $\tilde{b}$ such that $a \cap b_1 \neq \emptyset$, because a has a nonempty intersection with the upper cell of $\tilde{b}$ and this upper cell is a union of cells of the level of $\tilde{b}$. But by definition $\mathsf{NR}^{\mathsf{ap}}(c_1, \tilde{b})$. So $\mathsf{NR}(a, b)$. The argument for the other direction works similarly. □

The corresponding equivalent definition for NR within the framework of RCC (or more general: a region-based framework with a connectedness relation C) would be:

$$\mathsf{NR}(a, b) \text{ iff } \mathsf{C}(a, us(b^{\Uparrow})) \qquad (2)$$

The following example illustrates the nearness relations.

Example 2. *We define a partition chain with four levels as illustrated in Figure 1. Let $X = \{1, \ldots, 6\}$, $a_i = \{i\}$ for $1 \leq i \leq 4$ and*

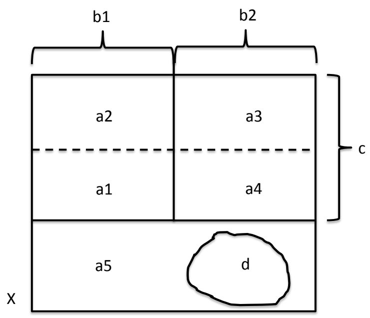

Figure 1. Illustration of configuration in Example 2

$a_5 = \{5, 6\}$. *All a_i are sets underlying cells of level* 0. *The set $d = 6$ is an arbitrary set (region) which does not underly any of these cells. The partitions on the levels 0 to 3 are defined by:*

- $\{(0, a_1), (0, a_2), \ldots, (0, a_5)\}$
- $\{(1, b_1), (1, b_2), (1, a_5)\}$ *where* $b_1 = a_1 \cup a_2$ *and* $b_2 = a_3 \cup a_4$.
- $\{(2, c), (2, a_5)\}$ *where* $c = b_1 \cup b_2$
- $\{(3, X)\}$.

It can be easily seen that $\mathsf{NR}(d, c)$, *because* $\tilde{c} = (2, c)$, *and* $c^{\Uparrow} = (3, X)$ *and* $d \cap X \neq \emptyset$. *Similarly* $\mathsf{NR}(d, (a_1 \cup a_4))$ *holds as* $\widetilde{a_1 \cup a_2} = (2, c)$. *It is not* $\mathsf{NR}(d, a_1)$ *as* $\tilde{a}_1 = (0, a_1)$ *and* $a_1^{\Uparrow} = (1, b_1)$ *but* $d \cap b_1 = \emptyset$. *Similarly one can see that not* $\mathsf{NR}(d, a_4)$.

4 PROXIMITIES AND NEARNESS

With proofs similar to the ones of [8] one can show that NR fulfills the properties of a *right-scaled proximity relation*. This is detailed out in the following proposition.

Proposition 2. *Let X be a set, pc be a partition chain over X and $\mathsf{NR}_{pc} = \mathsf{NR}$ be a nearness relation as defined by (1). The relation* NR *fulfills the properties of a right-scale scaled proximity, that is:*

1. *for all $a, b \subseteq X$: if $\mathsf{NR}(a, b)$, then a and b are nonempty;*
2. *for all $a, b, c \subseteq X$:*
   - *(a) if $\mathsf{NR}(a, b)$ or $\mathsf{NR}(a, c)$, then $\mathsf{NR}(a, b \cup c)$;*
     (if a is near one of b or c, then it is near the union of b and c.)
   - *(b) $\mathsf{NR}(a, c)$ or $\mathsf{NR}(b, c)$ if and only if $\mathsf{NR}(a \cup b, c)$;*
     (the union of a and b is near c iff one of the sets of the union a or b is near c.)
3. *if $a \cap b \neq \emptyset$, then $\mathsf{NR}(a, b)$.*
   (a and b have one element in common, then a is near b (and so also b is near a).)

The main difference of right-scaled proximities to minimal proximity structures in the meaning explicated by [1] is the fact that for right-scaled proximities the other direction in condition 2.(a) is, in general, not fulfilled, i.e., there may be sets a, b, c, such that a is near the union of b and c, formally: $\mathsf{NR}(a, (b \cup c))$, but neither is a near b nor is a near c. (Compare Ex. 2, where $\mathsf{NR}(d, a_1 \cup a_4)$ but neither $\mathsf{NR}(d, a_1)$ nor $\mathsf{NR}(d, a_4)$.) This is due to the fact that the union of b and c may belong to a higher level than b and c. So, putting two sets (in the second) argument together may have positive emergent effects—more concretely, the positive emergent effect of switching the level (or scale) from a lower to a higher one.

Note, that this kind of positive emergent effect is also handled by super-additive measures in general measure theory [12]. Classical measures μ have to be additive, i.e., must fulfill the condition that for disjoint events a, b we must have $\mu(a \uplus b) = \mu(a) + \mu(b)$. In generalized measure theory one considers measures that weaken this condition in both directions. μ is called super-additive iff $\mu(a \uplus b) \geq \mu(a) + \mu(b)$. It is called sub-additive iff $\mu(a \uplus b) \leq \mu(a) + \mu(b)$ [12, p.67]. Super-additivity means that the union has synergetic positive effects, sub-additivity means that the union has prohibiting effects.

It is possible to further characterise the case where a region is near a union of regions but not near one of them. Let δ denote a right-scaled proximity relation. Let a, b, c such $b \cap c = \emptyset$ and we have $\delta(a, (b \cup c))$ but not $\delta(a, b)$ and not $\delta(a, c)$. We call (b, c) an *irregular split* of $b \cup c$ w.r.t. a.

Definition 5 (Regularity)**.** *A weak right-scaled proximity δ over X is called* regular *iff for every set $a \subseteq X$ there is at most one irregular split of a set $b \cup c$ w.r.t. a.*

Now we can show that NR is a right-scaled proximity that fulfills the regularity condition.

Proposition 3. NR *is a regular right-scaled proximity relation.*

Proof. Assume $\mathsf{NR}(a, b \uplus c)$ and not $\mathsf{NR}(a, b)$ and not $\mathsf{NR}(a, c)$. As $us(b^{\Uparrow}) \cap a = \emptyset$ and $us((c^{\Uparrow}) \cap a = \emptyset$, we have $us(b^{\Uparrow}) \cup us(c^{\Uparrow}) \subsetneq us((b \uplus c)^{\Uparrow})$. We must have $us((b^{\Uparrow}) \neq us(c^{\Uparrow})$. Now, let $b \uplus c = b' \uplus c'$ where $b' \neq b$ and $c \neq c'$. One of b', c' must have elements of both b and c. W.l.o.g let us assume it is b'. That means that $\widetilde{b'} = \widetilde{b \cup c}$ and hence $\mathsf{NR}(a, b')$. □

Please note, that this property also holds for a model of the nearness relation NR which is defined in the RCC framework [9] according to (2). In this canonical model regions are defined to be regularly closed sets in the 2-dimension real plane. The crucial point is that the underlying sets b and c of cells that touch each other make up an irregular splitting of $b \cup c$ w.r.t. some region a—where $b \cup c$ stands for the sum operation of regions according to [9]. Now, one could move border points of b to c (or vice versa) in order to get a different irregular splitting $b' \cup c'$ of $b \cup c$ w.r.t. a; but b' and c' will not be regions anymore. Hence, the uniqueness of irregular splits is conserved, as long as b and c are constrained to be regions.

Another additional feature of the nearness relations NR_{pc} based on normal partition chains pc is that it fulfills the *connecting* property (cf. [1]), i.e., every region is near its complement or vice versa.

Proposition 4. *Let be given a normal partition chain pc and a nearness relation* $\mathsf{NR} = \mathsf{NR}_{pc}$ *according to the equivalent definition in (1). Then for all* $a \subseteq X$ *it holds that* $\mathsf{NR}(a, X \setminus a)$ *or* $\mathsf{NR}(X \setminus a, a)$.

Proof. Let $a \subseteq X$ be an arbitrary non-empty set. We have to show $\mathsf{NR}(a, X \setminus a)$ or $\mathsf{NR}(X \setminus a, a)$. First assume that a or $X \setminus a$ are not underlying sets of cells, e.g., w.l.o.g. assume a is not an underlying set of a cell. Then $us(\tilde{a})$ overlaps with $X \setminus a$ and we have $\mathsf{NR}(X \setminus a, a)$. Now assume that both a and $X \setminus a$ are (underlying sets of) cells. But, because the order is normal, either $a \subsetneq us(a^{\Uparrow})$ or $X \setminus a \subsetneq us((X \setminus a)^{\Uparrow})$, hence either $\mathsf{NR}(X \setminus a, a)$ or $\mathsf{NR}(a, X \setminus a)$. □

Note, that the proposition does not hold for arbitrary (i.e. non-normal) partitions chains as shown by the following example.

Example 3. *Assume* $a \neq \emptyset$. *We can construct a non-normal partition chain pc, such that in the first three levels one has the same two subsets* a_1, a_2 *as cells. That is, let* $X = a_1 \uplus a_2$ *and let for* $i \in \{0, 1, 2\}$ *be given the partition* a^i *of level i by* $((i, a_1), (i, a_2))$ *the partition of level 3 shall be* $(3, X)$. *Let* $\mathsf{NR} = \mathsf{NR}_{pc}$ *be the nearness relation defined by this non-normal partition chain. Then we have* $a_1 = X \setminus a_2$ *and* $a_2 = X \setminus a_1$ *but not* $\mathsf{NR}(a_1, a_2)$ *and not* $\mathsf{NR}(a_2, a_1)$.

In general, the nearness relations NR_{pc} for normal partion chains pc will not fulfill the so called strong axiom (3) for proximity relations δ (cf. [6]).

$$\begin{aligned} &\text{If not } \delta(a, b), \text{ there is an } e \subseteq X \text{ s.t.:} \\ &\quad \text{not } \delta(a, e) \text{ and not } \delta((X \setminus e), b) \end{aligned} \quad (3)$$

This axiom says that if a is not near b, there is a set e which separates a and b. In particular, if also for all sets a', b' with $a' \cap b' \neq \emptyset$ it holds that $\delta(a', b')$, then the fact that not $\delta(X \setminus e, b)$ entails $b \subseteq e$ (because it must be the case that $(X \setminus e) \cap b = \emptyset$).

We give a simple counterexample to the strong axiom.

Example 4. *Take* $X = \{1, 2, 3, 4, 5, 6\}$. *Consider the following normal chain as illustrated in Fig. 2:*

$$0 : \overbrace{\{1,2\}}^{a_1} \cup \overbrace{\{3,4\}}^{a_2} \cup \overbrace{\{5,6\}}^{a_3} \leq 1 : \overbrace{\{1,2\}}^{b_1} \cup \overbrace{\{3,4,5,6\}}^{b_2} \leq 2 : X$$

Take $a = \{1\}$, $b = a_2 = \{3, 4\}$. *Then not* $\mathsf{NR}(a, b)$. *But there is no* $e \supseteq b$ *such that not* $\mathsf{NR}(a, e)$ *and not* $\mathsf{NR}(X \setminus e, b)$. *The reason is: If* $e = b$, *then* $\mathsf{NR}(X \setminus b, b)$ *as* $(X \setminus b) \cap us(b^{\Uparrow}) \neq \emptyset$. *Similarly, if* $b \subsetneq e$, *then* $us(e^{\Uparrow}) = X$ *and hence* $\mathsf{NR}(a, e)$.

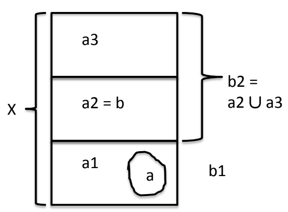

Figure 2. Illustration of configuration in Ex. 4

5 CELL PROPERTIES AND LOCAL DYNAMICS OF NEARNESS

In the subsections before we gave properties of the nearness relation NR that do refer only to NR but not to the underlying partition chain. As we will consider the effects of changing the arguments in NR and the effects of changing the partition chain on the induced nearness, we investigate in this section properties referring also to the partition chains. Concerning the first point of change, these properties are relevant to what we call the *local dynamics of nearness*. The investigation of local dynamics means—among other things—answering the following question: How does a change of the right argument of NR affect the set of sets considered near it? In particular, for which two regions (or more concretely: sets underlying cells b_1 and b_2) does the change from b_1 to b_2 conserve the nearness relations?

In order to answer (if only partly) this question, we introduce the following equivalence relations on the basis of a relation δ (which will be instantiated by NR) over a set X.

$$a^{\bullet} = \{b \subseteq X \mid \delta(a, b)\} \quad (4)$$
$$^{\bullet}a = \{b \subseteq X \mid \delta(b, a)\} \quad (5)$$
$$a \sim^{\bullet} b \quad \text{iff} \quad a^{\bullet} = b^{\bullet} \quad (6)$$
$$a \,^{\bullet}\!\!\sim b \quad \text{iff} \quad ^{\bullet}a = \,^{\bullet}b \quad (7)$$
$$a \sim b \quad \text{iff} \quad a \sim^{\bullet} b \text{ and } a \,^{\bullet}\!\!\sim b \quad (8)$$

As the identity $=$ is an equivalence relation (i.e., it is reflexive, symmetric, and transitive), the definitions immediately entail the fact that $\sim^{\bullet}, \,^{\bullet}\!\!\sim, \sim$ are equivalence relations, too. Now, if we look at cells (i, a) and (i, b) that are contained in the same upper cell, then these are left-equivalent.

Proposition 5. *Let* $a, b \subseteq X$ *such that* $\tilde{a} = (i, a)$, $\tilde{b} = (i, b)$ *and* $a^{\Uparrow} = b^{\Uparrow}$. *Then* $a \,^{\bullet}\!\!\sim b$.

Proof. Let $c \subseteq X$ be an arbitrary set. Then, by assumption $\mathsf{NR}(c, a)$ iff $\mathsf{NR}(c, b)$. □

Concerning the main question of the local dynamic of nearness this proposition has the following consequence: Changing the perspective from a cell to another cell of the same level with the same upper level does not change the perspective on what regions (as the first argument) are considered to be near. For illustration, consider again Fig. 1 in Example 2. Think of an agent that stays at cell a_1 and has calculated the regions near a_1. Then the agent moves to cell a_2, which has the same upper cell b_1. Then according to Prop. 5 he does not have update the regions near it as the regions near a_2 are exactly those near a_1. The situation is different if the agent moves from a_1 to a_4 which has a different upper cell than a_1.

A dual assertion with respect to this lemma is the observation that two disjoint cells are near each other in both directions iff they are cells on the same level with the same upper level cell.

Proposition 6. *For all sets* a, b *with* $\tilde{a} = (i, a)$ *and* $\tilde{b} = (j, b)$ *and* $a \neq b$ *the following equivalence holds:* $\mathsf{NR}(a, b)$ *and* $\mathsf{NR}(b, a)$ *iff* $i = j$ *and* $a^{\Uparrow} = b^{\Uparrow}$.

Proof. The direction from right to left follows from Prop. 5. For the other direction assume $\mathsf{NR}(a, b)$ and $\mathsf{NR}(b, a)$. Then by definition of NR, the first argument of the conjunct implies $a \cap us(b^{\Uparrow}) \neq \emptyset$. But this means, as a is the underlying set of a cell, that $a \subseteq us(b^{\Uparrow})$. As $a \cap b = \emptyset$, we can exclude the case that $a = us(b^{\Uparrow})$; hence, it follows that $us(a^{\Uparrow}) \subseteq us(b^{\Uparrow})$ and $i \leq j$. Symmetrically, we can deduce $us(b^{\Uparrow}) \subseteq us(a^{\Uparrow})$ and $j \leq i$. In the sum we get $b^{\Uparrow} = a^{\Uparrow}$. □

Moreover, if a, b are cells of the lowest level and are contained in the same upper level, then they are equivalent.

Proposition 7. *Let $a, b \subseteq X$ be such that $\tilde{a} = (0, a)$, $\tilde{b} = (0, b)$ and $a^{\Uparrow} = b^{\Uparrow}$. Then $a \sim b$.*

Proof. Because of Prop. 5 we are done with the proof if we can show that $a \sim^{\bullet} b$. Let $c \subseteq X$ be an arbitrary set. $\mathsf{NR}(a, c)$ iff (by definition) $a \cap us(c^{\Uparrow}) \neq \emptyset$ iff (as different cells are either disjoint or comparable with respect to $\subseteq$, and the level of $\tilde{c}$ is greater than or equal to the level of $\tilde{a}$) $us(a^{\Uparrow}) \subseteq us(c^{\Uparrow})$ iff $us(b^{\Uparrow}) \subseteq us(c^{\Uparrow})$ iff $\mathsf{NR}(b, c)$. □

Again, concerning the main question of the local dynamic of nearness this proposition has the following consequence: Changing the perspective from a cell to another cell on the lowest level, where both have the same upper level, does not change the set of regions that are considered to be near—and this holds in both cases of changing the first argument or of changing the second argument.

6 MERGING AND GLOBAL DYNAMICS OF NEARNESS

In their study of regional changes of municipalities in Finland, Kauppinen and colleagues [5] found seven kinds of type changes which are as follows:

1. a region is established
2. two or more regions are merged into one
3. a region is split into two or more regions
4. a region's name is changed
5. a region is annexed to a different country
6. a region is annexed from a different country
7. a region is moved to another city or municipality

We are interested in changes that concern changes of cells for partitions in a given partition chain. Hence we adapt a subset of the types of changes to our setting by explicitly formalizing the type of change.

Clearly the most interesting changes are that of merging two regions to a new region and its counterpart, the split of regions into two regions. These types of changes are low frequent-changes (in contrast to the local dynamics case where an agent updates the nearness relations when moving around); e.g., Kauppinen and colleagues [5] recognized 144 merges and 94 splits of municipalities in Finland between 1865 and 2007. But nonetheless, the effects of merges and splits on the nearness relation are worth to be investigated.

Here, we restrict our attention to different forms of merging. We have to explain what it means that two cells (of a partition) are merged, and whether such a merge is possible such that the result is again a (normal) partition chain.

So let pc be a normal partition chain over X having levels 0 to n. We will look at merging two cells on the same level into a new cell; in order to get a first rough picture on the effects of merging, we look at the special case where the cells are members of the next-to-last level $n-1$. In this case, both cells to be merged have always the same upper cell, namely X. For illustration of the possible merge operations have a look at the partition chain in Fig. 3, which we have arranged such that one can see the tree structure of the the partition chain, with X being its root. The cells labelled with the letter c make up the cells of the next-to-last level 2. The different forms of changes within a partition chain can be seen as different forms of updating a tree.

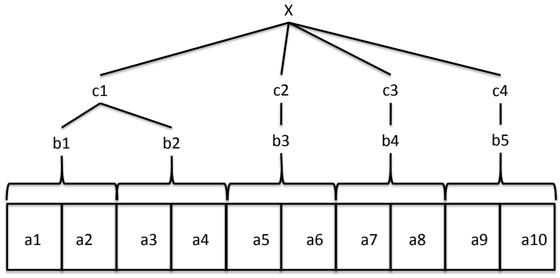

Figure 3. Illustration of example configuration for merge

Merging the cells $(2, c_2)$ and $(2, c_3)$ into a new cell means that the underlying set of the merging result has to have the union of c_2 and c_3 as the underlying set. But there are in principle two ways to conduct this merge that depend on specifying the level of the merge result.

The first option is to modify the next-to-last level, so that the whole number of levels is untouched. In case of the example illustrated in Fig. 3 this would mean that the partition of c-cells is substituted by the new partition of c-cells that consists of the cells $(2, c_1)$, $(2, c_2 \cup c_3)$ and $(2, c_4)$ (see Fig.4). We term this type of merge *level modifying merge—lm merge* for short. If a normal partition chain pc_2 results from another normal partition chain pc_1 by an lm merge, then we write $pc_1 \leadsto^{lm} pc_2$.

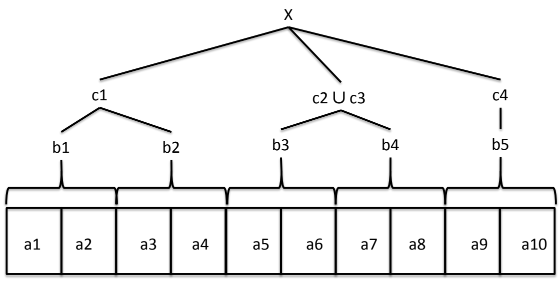

Figure 4. Illustration of merge by modifying

The other option is to make the union of the sets to be part of a new level. Hence, in addition to the original partition made up by $(2, c_1)$, $(2, c_2)$, $(2, c_3)$ and $(2, c_4)$, one adds the partition $(3, c_1)$, $(3, c_2 \cup c_3)$ and $(3, c_4)$ and raises the level of X by one to $(4, X)$ (see Fig. 5). We term this type of change *level adding merge—la merge* for short. If a normal partition chain pc_2 results from another normal partition chain pc_1 by a la merge, then we write $pc_1 \leadsto^{la} pc_2$.

In some cases, either form of merge may not be possible without violating the normality condition. For example, if the next-to-last level consists only of two cells $(n-1, x_1)$ and $(n-1, x_2)$, then the union of x_1 and x_2 is the whole domain X; so the merge results in the same set partition (X) on two different levels, which violates the normality condition.

What can we say about the change of the nearness relation induced by level modifying merges on the next-to-last level? First we note

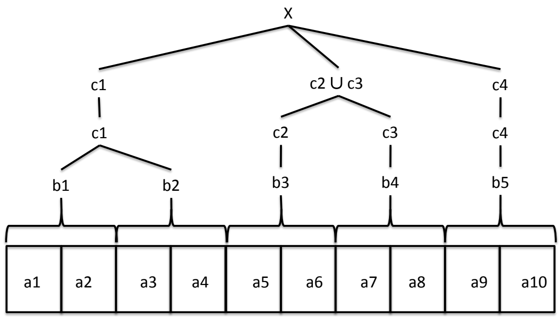

Figure 5. Illustration of merge by adding

that the level of a set in pc_1 is identical to the level in pc_2 if the former is below or equal to $n-1$. If its level in pc_1 is n, then its level in pc_2 may be n or $n-1$.

The change of pc_1 into pc_2 affects only the next-to-last partition, e.g., by merging cells $(n-1, c_1)$ and $(n-1, c_2)$; hence, the nearness relation is affected only locally. So, if the second argument b has level at most $n-3$, then one can say that a is near b in pc_2 if and only if it is near in pc_1.

Proposition 8. *Let pc_1, pc_2 be two normal partition chains over X such that $pc_1 \rightsquigarrow^{lm} pc_2$ w.r.t. cells $(n-1, c_1)$ and $(n-1, c_2)$ on the next-to-last level $n-1$. Then the following assertions hold:*

1. *For all sets $a \subseteq X$ and all sets $b \subseteq X$ with level $l_{pc_2}(b) \leq n-3$ one has : $\mathsf{NR}_{pc_1}(a, b)$ iff $\mathsf{NR}_{pc_2}(a, b)$.*
2. *For all sets $a, b \subseteq X$: If $\mathsf{NR}_{pc_1}(a, b)$, then $\mathsf{NR}_{pc_2}(a, b)$.*

Proof. The assertions can be proved as follows:

1. This assertion follows from the fact, that for all $b \subseteq X$ with level at most $n-3$ (in pc_2) the upward cells in both pc_1 and pc_2 are identical, $b^{\Uparrow, pc_1} = b^{\Uparrow, pc_2}$. Hence, by definition of nearness it immediately follows that $\mathsf{NR}_{pc_1}(a, b)$ iff $\mathsf{NR}_{pc_2}(a, b)$.
2. In order to proof this assertion suppose $\mathsf{NR}_{pc_1}(a, b)$, i.e., $a \cap us(b^{\Uparrow, pc_1}) \neq \emptyset$. We distinguish different cases depending on the level $l_{pc_1}(b)$ of b in pc_1.
Assume $l_{pc_1}(b) = n-2$, then $b^{\Uparrow, pc_1} = (n-1, c)$ for some set c on the level $n-1$. If $c = c_1$ or $c = c_2$, then $b^{\Uparrow, pc_2} = (n-1, c_1 \cup c_2)$. So from $a \cap us(b^{\Uparrow, pc_1}) \neq \emptyset$ one deduces $a \cap us(b^{\Uparrow, pc_2}) \neq \emptyset$, i.e. $\mathsf{NR}_{pc_2}(a, b)$. If c is an underlying set of another cell on level $n-1$, then we have $us(b^{\Uparrow, pc_2}) = (n-1, c)$ and hence also $\mathsf{NR}_{pc_2}(a, b)$.
Now assume that $l_{pc_1}(b) = n-1$. Then $\tilde{b}^{pc_1} = (n-1, c)$ for some set c on the partition level $n-1$. Then we will have $\tilde{b}^{pc_2} = (n-1, c')$ for $c \subseteq c'$. Hence, $b^{\Uparrow, pc_2} = (n, X)$ and so $\mathsf{NR}_{pc_2}(a, b)$.
Last assume that $l_{pc_1}(b) = n-1$. In this case, the level of b in pc_2 may be $n-1$ or n. But in any case, one has $b^{\Uparrow, pc_2} = (n, X)$, and therefore $\mathsf{NR}_{pc_2}(a, b)$.

□

The consequence of this proposition for a cognitive agent using NR as a nearness notion is that it has to update his NR graph only locally when the partition chain is updated by a level modifying change.

Due to the level addition, the situation for la merges is a little bit different. For example, considering our example partition chain illustrated in Fig. 3 one can have $a \subseteq X$ such that $\mathsf{NR}_{pc_1}(a, c_2)$ but not $\mathsf{NR}_{pc_2}(a, c_2)$ because, the upper level cell of $(n-1, c_2)$ in pc_1 is the biggest cell (n, X), but in pc_2 the upper cell is $(n-1, c_2 \cup c_3)$. So, choosing, e.g., $a = a_1$ and $b = c_2$ one has $\mathsf{NR}_{pc_1}(a, b)$ but not $\mathsf{NR}_{pc_2}(a, b)$. But still we can show as above that sets with level below $n-3$ have the same nearness relations.

Proposition 9. *Let pc_1, pc_2 be two normal partition chains over X such that $pc_1 \rightsquigarrow^{la} pc_2$ w.r.t. cells $(n-1, c_1)$ and $(n-1, c_2)$ on the next-to-last level $n-1$. Then for all sets $a \subseteq X$ and all sets $b \subseteq X$ with level $l_{pc_2}(b) \leq n-3$ one has : $\mathsf{NR}_{pc_1}(a, b)$ iff $\mathsf{NR}_{pc_2}(a, b)$.*

7 CONCLUSION

Cognitive agents using a hierarchical nearness relation based on a partition chain have to deal with two aspects of dynamics of nearness, the local dynamics (the cognitive agent changes his position and so has to update the nearness relations) and a global dynamics (the partition chain may change, and hence the induced nearness relation has to be changed.) We have shown that under some circumstances both a local change and a global change affect the nearness relation only w.r.t. a small set of regions; hence, under these circumstances, the nearness relations between few regions have to be updated.

We gave preliminary results on the local dynamics and on the global dynamics of the partition-chain based nearness relation. The results on global dynamics have to be completed by investigations on merges for levels below the next-to-last level. In this case one will have to differentiate between merging regions with the same upper level cells vs. merging regions with different upper level cells. Additionally one has to define how to propagate the merge effect to the higher levels (as the merger on level i may affect also cells on levels above $i+1$.) Moreover we plan to define adaptations of the other changes mentioned by [5] to the partition-chain framework and investigate their effects on the change of the nearness relation.

The presented approach considers only partition chains, i.e. a totally ordered set of nested partitions. For more realistic approaches we are going to formally investigate the more general scenario where partitions may not be nested/aligned. This is, e.g., the case when one considers micro functional regions [2] in addition to administrative units.

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