# Investigation of Sensor Networks using Algebraic Topology

## Bachelor Thesis

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# 1 Introduction

Sensor networks are systems appearing in real life, which measure physical quantities using sensor nodes and convert them to a signal. Reading and analysing these signals is a task, which can be done using the algebraic topology as a mathematical aid. This thesis is going to introduce general concepts of algebraic topology in the context of abstract simplicial complexes and show which tools can be used in the field of sensor networks.

First, in chapter 2 some mathematical fundamentals have to be discussed, so that chapter 3 can then introduce the Vietoris-Rips complex and describe an application for it in a real sensor network. Finally, a conclusion and an outlook for possible future work is given in chapter 4.

With respect to definitions appearing, this thesis closely follows the work of Kozlov [1], Ghrist [2, 5] and Zimmermann [3].

# 2 Mathematical Basics

In order to understand the idea of sensor networks and the underlying topological spaces, some fundamental mathematical terms and ideas of algebraic topology have to be explained first. This chapter will give the reader an overview about these mathematical basics.

## 2.1 Abstract Simplicial Complexes

The topological spaces that occur in this thesis are to be described in a simple way as the following.

**Definition 1** *Let $A$ be a finite set and $\Delta$ a collection consisting of subsets of $A$. Then $\Delta$ is called an abstract simplicial complex if $X \in \Delta$ and $Y \subseteq X$, then $Y \in \Delta$.*

For brevity purposes the abstract simplicial complexes from Definition 1 are sometimes just called *simplicial complexes*. They are denoted by the symbol $\Delta$. The elements $v \in \mathrm{A}$ or also $v \in \Delta$ are called *vertices* of $\Delta$ (singular: *vertex*). The set of all vertices of $\Delta$ shall be $V(\Delta)$.
If all subsets of A are included in $\Delta$, it is denoted by $\Delta^A$ and called a *simplex*. The sets $\sigma \in \Delta$ are called *simplices*. Unique simplices $\sigma \in \Delta$ that are not part of any other simplex of $\Delta$ are *maximal*.

Figure 1 shows some examples of simplices, which are not simplicial complexes, because they violate one of the conditions above. The left simplex is missing a vertex in the middle of the two edges, so that four edges would be generated. For the middle simplex an edge between the two unconnected vertices needs to be added in order to form a triangle. In the last example the triangle is crossed by one edge at an interior point.

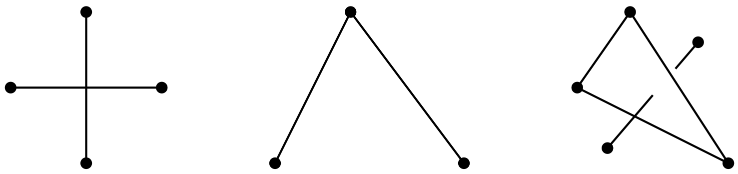

Figure 1: Non-examples of simplicial complexes

**Example 1** *The collection of sets {{0, 1}, {0, 2}, {1, 2}, {0}, {1}, {2}, ∅} forms the simplicial complex* $\Delta$. *But in order to get a simplex* $\Delta^{\{0,1,2\}}$ *the set {0, 1, 2} would have to be added to the collection.*

Simplices have a *dimension* , which is denoted as dim $\Delta$ in the following. The dimension is defined as the maximum cardinality of the elements of the simplex. If a simplex $\Delta$ has the dimension dim $\Delta = k$, it can be called a $k$-simplex. As the empty set has the cardinality 0, its dimension equals -1. Therefore in an Euclidean space $\mathbb{R}^n$ for all $-1 \leq k \leq n$, with n being an integer, there is a simplex of the dimension n.
From left to right figure 2 shows simplices from the dimension -1 to 3 and thus starting from the (-1)-simplex to the 3-simplex there are five simplices of the $\mathbb{R}^3$: empty set, vertex, edge, triangle and tetrahedron.

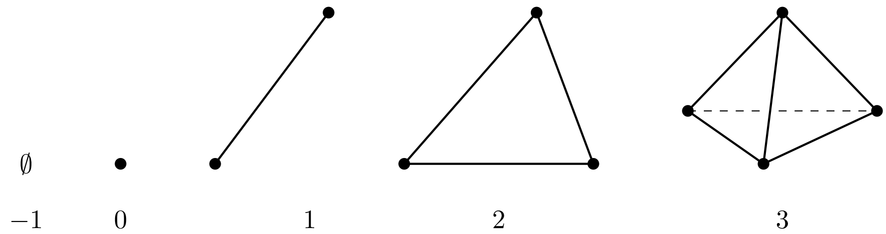


Figure 2: Simplices in $\mathbb{R}^3$ with rising dimension, denoted below each simplex

Each $k$-simplex has $k+1$ points (or vertices). A convex hull of any nonempty subset of these $k+1$ points is called a *face* of the simplex. Thus each face is a simplex itself. Let $\Delta_1$ and $\Delta_2$ be simplices and $\Delta_1$ a face of $\Delta_2$. Furthermore if dim $\Delta_1 = l$, then $\Delta_1$ is called an $l$-face. The 0-faces are vertices, the 1-faces edges and so on.
There are two *improper faces* of $\Delta_2$, the empty set $\Delta_1 = \emptyset$ and the original simplex itself $\Delta_1 = \Delta_2$. All other faces of $\Delta_2$ are *proper*. For each simplex $\Delta$ the number of $l$-faces of a $k$-simplex can be easily calculated by determining the number of ways $l+1$ can be chosen from $k+1$ points:

$$\binom{k+1}{l+1} = \frac{(k+1)!}{(l+1)!(k-l)!}.$$

And so the total number of faces in all dimensions is

$$\sum_{l=-1}^{k} \binom{k+1}{l+1} = 2^{k+1}.$$

Let $\Delta_1, \Delta_2$ be abstract simplicial complexes, then $\Delta_1$ is an *abstract simplicial subcomplex* of $\Delta_2$ if $v \in \Delta_1$ implies $v \in \Delta_2$, which is denoted as $\Delta_1 \subseteq \Delta_2$. Additionally if there exists a $v \in \Delta_2$ such that $v \notin \Delta_1$, $\Delta_1$ is a *proper subcomplex* of $\Delta_2$.

For abstract simplicial complexes the dimension is equal to the maximum dimension of its simplices. In this regard, if $\Delta_1$ is an abstract simplicial subcomplex of $\Delta_2$, then dim $\Delta_1 \leq$ dim $\Delta_2$.

**Definition 2** *If an n-simplex has a specified orientation, it is called an oriented n-simplex. The orientation is a collection of orderings for each vertex in the set.*

Any $n$-simplex has $(n+1)!$ distinct ordered simplices, of which not all should be distinguished between. There are only two geometrically different orientations, clockwise and anti-clockwise.

**Example 2** *The triangle, a 2-simplex, has six different associated ordered simplices. The denotations* $[v_0, v_1, v_2]$, $[v_1, v_2, v_0]$, $[v_2, v_0, v_1]$ *order the triangle in the one direction, while* $[v_0, v_2, v_1]$, $[v_2, v_1, v_0]$, $[v_1, v_0, v_2]$ *order them in the other direction. Thus*

$$[v_0, v_1, v_2] = [v_1, v_2, v_0] \neq [v_2, v_1, v_0].$$

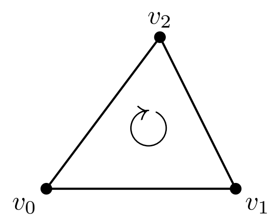


Figure 3: Oriented 3-simplex

**Example 3** *In figure 3 an oriented complex has the following set of vertices* $[v_0, v_2, v_1]$. *The set* $[v_0, v_2, v_1]$ *has the dimension* $d = 2$ *and is therefore called a 2-simplex.*

*In this example the simplicial complex is as following:*

$$\Delta = \{[v_0, v_1, v_2], [v_0, v_2], [v_2, v_1], [v_1, v_0], [v_0], [v_1], [v_2], \emptyset\}$$

*In this case the set* $[v_1, v_0]$ *is a 1-face and* $[v_0]$ *is a 0-face. In general it can be said that a set containing n vertices is an n-face.*

Given an abstract simplicial complex $\Delta$ and a field $\mathrm{K} = \mathbb{R}$ a real space can be generated as $C_n = C_n(\Delta, \mathbb{R})$ with its own basis elements.
These elements are given by oriented $n+1$ - simplices $[v_0, \dots, v_n]$ and the relations for the simplices $[v_0, \dots, v_n] = (-1)^{sgn(\pi)} \cdot [v_{\pi(1)}, \dots, v_{\pi(n)}]$, where $\pi$ is a permutation of the degree n+1.

**Example 4** *The following set of edges $C_1 = \{[v_0, v_2], [v_2, v_1], [v_1, v_0]\}$ are possible basis elements of the space $\mathbb{R}^3$.*
*Any edge, like in this case $[v_0, v_2]$, can also be read in the other direction, additionally including a negative sign and thus $[v_0, v_2] = -[v_2, v_0]$. Also in this case the permutation is $\pi = (0, 2)$ and therefore $sgn(\pi) = -1$.*
*In order to span the whole $\mathbb{R}^3$-space the basis elements have to be multiplied by real numbers like*

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \cdot [v_0, v_2] + b \cdot [v_2, v_1] + c \cdot [v_1, v_0], \quad \text{where a,b,c } \in \mathbb{R}.$$

*Accordingly a real space can also be described by the vertices or the triangle of the oriented complex, so that for all $C_n(\Delta)$ in $\mathbb{R}^3$:*

$$\begin{aligned} C_0 &= \langle [v_0], [v_1], [v_2] \rangle = \mathbb{R}^3, \\ C_1 &= \langle [v_0, v_2], [v_2, v_1], [v_1, v_0] \rangle = \mathbb{R}^3, \\ C_2 &= \langle [v_0, v_1, v_2] \rangle = \mathbb{R}^1. \end{aligned}$$

## 2.2 Operators and Maps

**Definition 3** *The boundary operator $\delta_n : C_n \to C_{n-1}$ is a linear transformation, which is defined as*

$$[v_0, \dots, v_n] \mapsto \sum_{k=0}^{n} (-1)^k \cdot [v_0, \dots, \hat{v}_k, \dots, v_n].$$

The operator $\delta_n$ maps a set of basis elements of the $n$-th dimension to a sum of elements of $(n-1)$-th dimension by taking out $\hat{v}_k$.

**Example 5** *The 3-simplex shown in figure 4, a tetrahedron, is mapped to four different triangles by $\delta_3 : C_3 \to C_2$. The result is a sum of these four triangles with alternating signs, each with one of the vertices taken out as*

$$\begin{aligned} \delta_3([v_0, v_1, v_2, v_3]) &= \sum_{k=0}^{3} (-1)^k \cdot [v_0, \dots, \hat{v}_k, \dots, v_3] \\ &= [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]. \end{aligned}$$

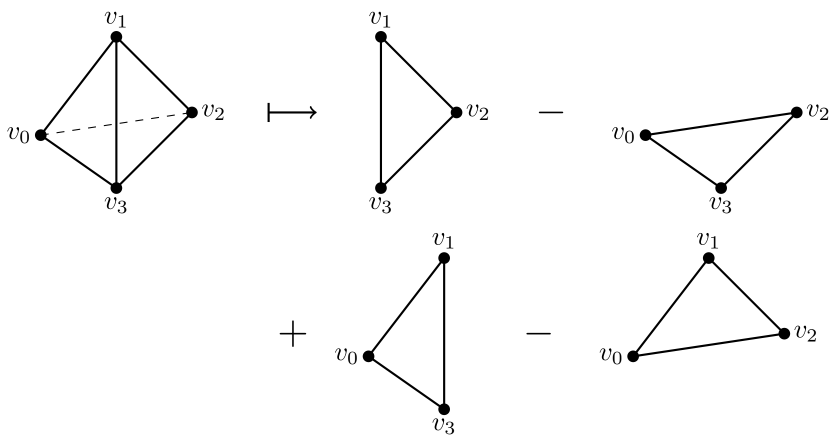


Figure 4: Boundary operator applied to a tetrahedron

A triangle (2-simplex) is mapped to its three edges, which it consists of:

$$\delta_2 : C_2 \to C_1 : [v_0, v_1, v_2] \mapsto [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$

The map $\delta_n$ is linear. This can easily be shown by linear extension, as $\delta_n$ is defined on basis elements and can be linearly extended.
Furthermore it is known that for each n $\geq 0$,

$$\delta_n \circ \delta_{n+1} = 0, \tag{1}$$

meaning that each boundary of a chain is an empty boundary.
A proof for that statement can be found in [1, page 40]. Here it will be illustrated using a graphic, see Figure 5. The left ellipse shows the space $C_{n+1}$, from which the operator $\delta_{n+1}$ maps to $C_n$. This time though only a part of the space $C_n$ is occupied by the newly generated simplex, shown by the gray area. Ultimately the simplices are mapped to 0 in the $C_{n-1}$ space by $\delta_n$, seen at the right ellipse.

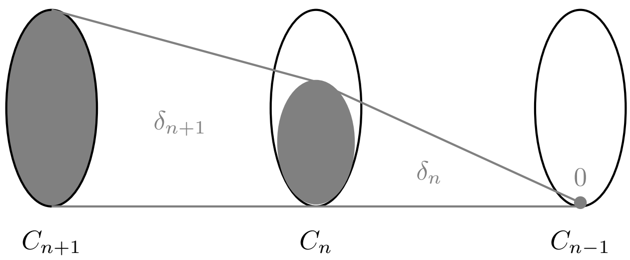


Figure 5: Boundary of a chain complex

**Example 6** *This example will demonstrate that the boundary of a chain is 0. For that, the tetrahedron from the example above will be used again. It was already mapped to its triangles and is now further mapped to its edges, thus being mapped from* $C_3$ *to* $C_1$*:*

$$\delta_3([v_0, v_1, v_2, v_3]) = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

*and by applying the second boundary operator*

$$\begin{aligned} \delta_2(\delta_3([v_0, v_1, v_2, v_3])) &= -[v_2, v_3] + [v_1, v_3] - [v_1, v_2] - (-[v_2, v_3] + [v_0, v_3] - [v_0, v_2]) + \\ &\quad (-[v_1, v_3] + [v_0, v_3] - [v_0, v_1]) - (-[v_1, v_2] + [v_0, v_2] - [v_0, v_1]) \\ &= 0. \end{aligned}$$

Using the boundary operator multiple times results in a chain complex, which is a sequence of vector spaces connected by linear transformations:

$$\cdots \xrightarrow{\delta_{n+2}} C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} 0.$$

The image of $\delta_{n+1}$ is

$$\mathrm{Im}(\delta_{n+1}) = \{\delta_{n+1}(v) \mid v \in C_{n+1}\}.$$

And the kernel of $\delta_n$ is

$$\mathrm{Ker}(\delta_n) = \{v \in C_n \mid \delta_n(v) = 0\}.$$

The image of the boundary operator $\delta_{n+1}$ always lies in the kernel of $\delta_n$, which means $\mathrm{Im}(\delta_{n+1}) \subseteq \mathrm{Ker}(\delta_n)$. This follows immediately from equation (1).

Both $\mathrm{Im}(\delta_{n+1})$ and $\mathrm{Ker}(\delta_n)$ are subspaces of $C_n$. As a result there are two subspaces of $C_n$, which are different from each other. They are defined as the following:

$$n - \text{cycles} : Z_n(\Delta) = \mathrm{Ker}(\delta_n),$$
$$n - \text{boundaries} : B_n(\Delta) = \mathrm{Im}(\delta_{n+1}).$$

Because of equation (1), for all $n \geq 0 : B_n(\Delta) \subseteq Z_n(\Delta)$.
The $n$-cycles of the simplicial complex $\Delta$ are the basis elements counting the holes of dimension $n$ of $\Delta$, denoted by the Betti number, which will be described in the next subsection.

## 2.3 Homology Groups

Let V be a vector space in $\mathbb{R}$ and U a subspace of V. The *quotient vector space* is then defined as $V \big/ U := \{v + U \mid v \in V\}$.
Figure 6 shows U as a subspace of V, in this case U is a line through the origin in the

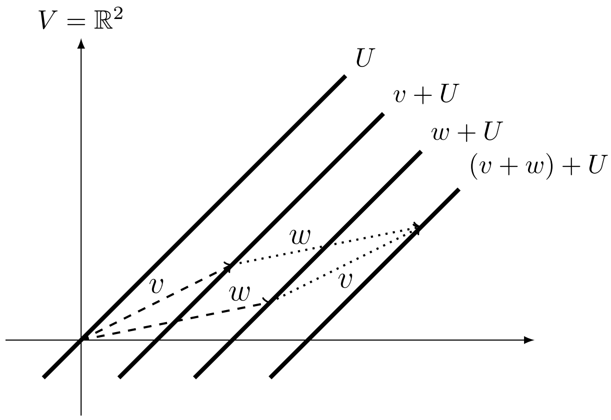


Figure 6: Subspaces U, V in $\mathbb{R}^2$

space $\mathbb{R}^2$. Affine subspaces are reached by adding the vectors v or w (dashed arrows), which act like vectors in $\mathbb{R}^2$. Adding the sum of v and w can also be displayed by adding either vector after the other. This is displayed by the dotted arrows.

The equivalence classes are $\overline{v} = \{w \in V \mid v \sim w\}$ and as shown in figure 6 $v + U = \{v + u \mid u \in U\}$, which is an affine subspace of U. Then $\overline{v} = v + U$. According to the linearity of the relation:

$$\begin{aligned} \overline{v} + \overline{w} := \overline{v+w} \quad &\Leftrightarrow \quad (v+U) + (w+U) = (v+w) + U \\ s \cdot \overline{v} = \overline{s \cdot v} \quad &\Leftrightarrow \quad s \cdot (v+U) = (s \cdot v) + U \quad, s \in \mathbb{R}. \end{aligned}$$

**Definition 4** *The quotient vector space of the cycles and boundaries is defined as an n-dimensional homology group*

$$H_n(\Delta) = Z_n(\Delta) \Big/ B_n(\Delta).$$

*If the difference of two cycles $v, w \in Z_n(\Delta)$ is a boundary, the cycles are called homologous or equivalent, as described by the relation*

$$v \sim w \quad \Leftrightarrow \quad v - w \in B_n(\Delta).$$

For the dimension of the quotient vector space it is known that

$$\dim\, V = \dim\, U + \dim\, V \Big/ U.$$

With $\dim\, V \Big/ U = 1$, then $\dim\, V > \dim\, U$. This yields for the dimension of $H_0(\Delta)$ the number of connected components of $\Delta$.

Let $[v]$ be the denotation for any element of $C_0(\Delta) = Z_0(\Delta)$ and $\overline{[v]} \in H_0(\Delta)$ be the class of this element. Then

$$\overline{[v]} = [v] + U = [v] + B_0(\Delta).$$

Now let $[v]$ and $[w]$ be two vertices of $C_0(\Delta)$ and $[v, w]$ an edge of $\Delta$, then with

$$\delta_1([v, w]) = [w] - [v] = B_0(\Delta),$$

it can be concluded that

$$\begin{aligned} [v] + B_0(\Delta) = [v] + ([w] - [v]) + B_0(\Delta) = [w] + B_0(\Delta) \\ \Rightarrow \overline{[v]} = \overline{[w]}. \end{aligned}$$

This means that the two elements from $H_0(\Delta)$ have the same class. This is also true for the vertices in any connected component.

$H_n(\Delta)$ is the *n-dimensional homology* of $\Delta$. The *Betti number* is denoted by $\beta$ and exists for every homology group $H_*(\Delta)$ so that each dimension has its own Betti number defined as $\beta_n(\Delta) := \dim(H_n)$. As stated before the $n^{th}$ Betti number counts the number of holes in $H_n$ by counting the number of $n$-cycles not corresponding to the $n+1$-boundaries. The first three Betti number are intuitive and conceivable. $\beta_0$ counts 1-dimensional holes, which are the connected components. $\beta_1$ is the number for the 2-dimensional or circular holes. And lastly $\beta_2$ counts the 3-dimensional holes called cavities or voids.

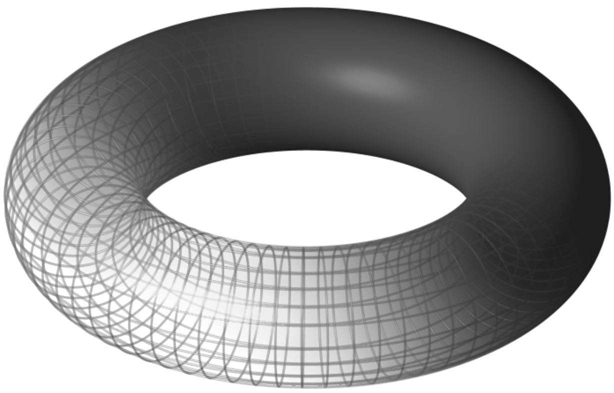

Figure 7: 3-dimensional hollow torus, picture taken from [7]

**Example 7** *Figure 7 shows a 3-dimensional torus, which has the following Betti numbers*

$$\beta_0 = 1,$$
$$\beta_1 = 2,$$
$$\beta_2 = 1.$$

*This is because the torus is one wholly connected component ($\beta_0$), which has two circular holes: The one in the middle and one in the inside of the torus ($\beta_1$). Also the inside of the tube is a 3-dimensional cavity ($\beta_2$).*

**Example 8** *Let the abstract simplicial complex $\Delta$ be a triangle with the set of vertices $[v_0, v_1, v_2]$ and the set of edges $\{[v_0, v_1], [v_1, v_2], [v_0, v_2]\}$. The two relevant chain complexes for this example are*

$$C_0 = \langle [v_0], [v_1], [v_2] \rangle = \mathbb{R}^3,$$
$$C_1 = \langle [v_0, v_1], [v_1, v_2], [v_0, v_2] \rangle = \mathbb{R}^3,$$

*and the boundary operator $\delta_1([v_i, v_j]) = -[v_i] + [v_j]$, for all $0 \leq i, j \leq 2$ and $i \neq j$. The homology groups then are*

$$H_0(\Delta) = Z_0(\Delta) \big/ B_0(\Delta) = C_0(\Delta) \big/ \langle [v_0] - [v_1], [v_0] - [v_2] \rangle = \langle v_0 + B_0(\Delta) \rangle = \mathbb{R},$$
$$H_1(\Delta) = Z_1(\Delta) \big/ B_1(\Delta) = Z_1(\Delta) = \langle [v_0, v_1] - [v_1, v_2] + [v_0, v_2] \rangle = \mathbb{R},$$

*and the higher dimensional homology groups are $H_i(\Delta) = 0$ for $i > 1$.*

# 3 Sensor Networks

For the application of algebraic topology in sensor networks a few more mathematical ideas have to be explained. These ideas can be directly used to describe a network of nodes as a simplicial complex.
Often in real-life applications data is gathered and represented as an unordered sequence of points in a real Euclidean space $\mathbb{R}^n$ of dimension $n$. For this thesis it is convenient to assume that the data comes from a series of sensor readings. The data is then called *point cloud data*. Obtaining the cloud of points from the physical sensors is easy and the goal is now to topologically represent the data of the sensors.

This can be done by converting the point cloud into vertices of a graph. The connections of the points would be illustrated by edges and determined by a specified distance between the vertices, which has to be met. Even though this is a rather primitive approach, the underlying idea remains the same for a higher dimensional representation, which also grasps the more complex structures of point clouds.

## 3.1 Vietoris-Rips Complex

The *Vietoris-Rips complex* (often abbreviated as Rips complex or Vietoris complex) is an abstract simplicial complex in the Euclidean space $\mathbb{R}^n$.

**Definition 5** *Let $V$ be a set of points (or vertices) in $\mathbb{R}^n$. With a non-negative real number $\epsilon$ the Vietoris-Rips complex is denoted as $\mathcal{R}_\epsilon(V)$. Its simplices are formed on every finite set of points within the pairwise distance $\epsilon$. Each subset of $V$ of $(k+1)$ points forms a $k$-simplex, which is included in the Vietoris-Rips complex.*

The generation of a sample complex is illustrated in Figure 8. In this case V is a set of nine points (red) around which circles with the radius $\epsilon/2$ (diameter $\epsilon$) are drawn. Those circles indicate the maximum distance another point can have from the first point with them having a connection. A pair is formed by the two points, which is represented by an edge (black). For this, overlapping circles indicate a connection between two or more points. Triples of points are shown by the three yellow triangles and lastly there is also one quadruple of points in green.

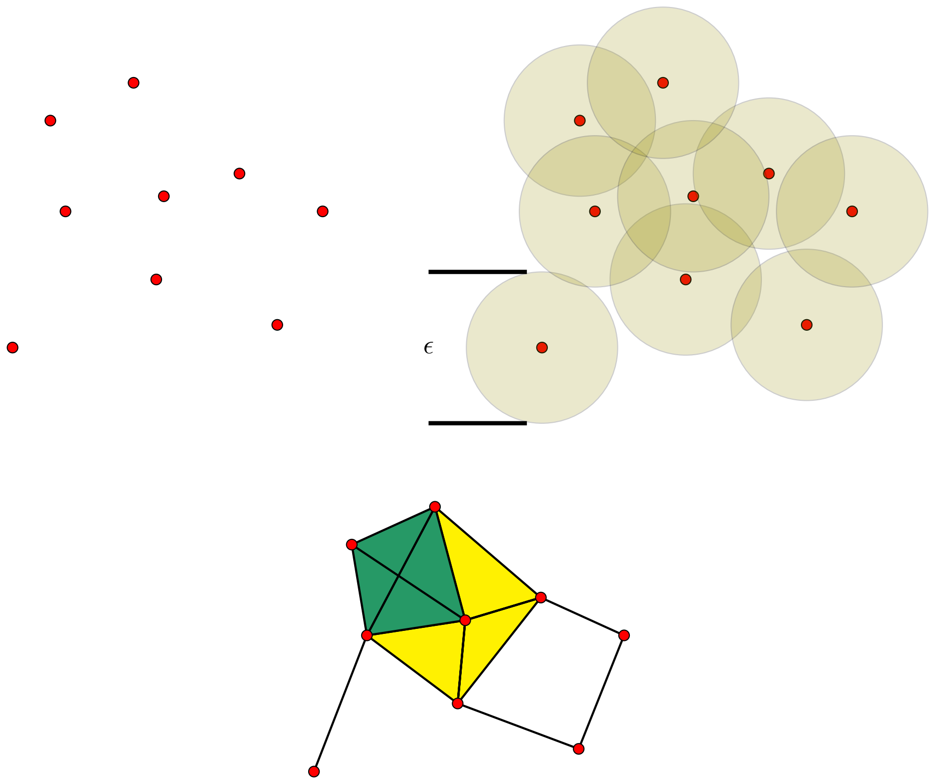


Figure 8: Step-by-step generation of a Vietoris-Rips complex

Choosing a good value for $\epsilon$ is an essential task, which can be very difficult. For an extremely small $\epsilon$ the complex is a discrete set, without any connections between the points. A very large $\epsilon$ results in only one simplex, which has a very high dimension. Both outcomes are undesirable, as they do not capture the underlying topology of the data set in most cases. Therefore a suitable $\epsilon$ needs to be found for a good approximation. To illustrate the ramifications of a smaller or larger $\epsilon$ it is best to revisit the complex from Figure 8.

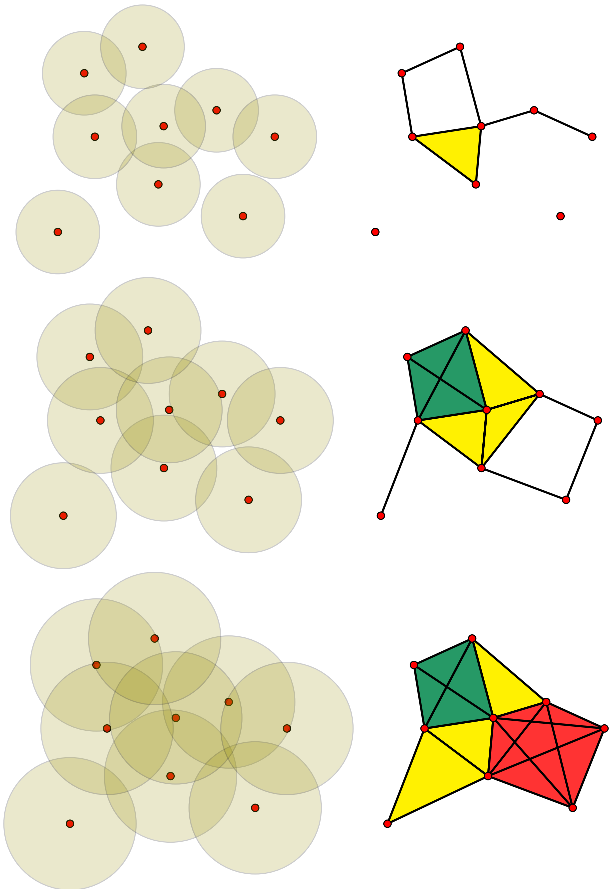

Figure 9: A series of Vietoris-Rips complexes with an increasing $\epsilon$

In the first complex of Figure 9, $\epsilon$ starts up smaller than in Figure 8. This results in only one triple of points, but more importantly two points are completely disconnected from the rest, as the $\epsilon$ was too small for them to form pairs. The parameter $\epsilon$ is then gradually increased, which results in the Vietoris-Rips complex already known from Figure 8. New pairs are formed and the hole in the upper left corner has turned into a quadruple of points. However a new hole has occurred in the bottom right corner.

The last graphic shows the complex for a further increase of $\epsilon$. The most significant change is the hole, which has turned into a quintuple of points (red).
If $\epsilon$ would be increased even further, some of the outer points would make connections, thus covering even more space. On the downside, the dimension of several triplets or quadruples would also increase, making the complex more complicated and difficult to compute in reality.

## 3.2 Relation to Čech Complex

Like the Vietoris-Rips Complex the Čech Complex is an abstract simplicial complex in $\mathbb{R}^n$.

**Definition 6** *Let $V$ be a set of points (or vertices) in $\mathbb{R}^n$. With a non-negative real number $\epsilon$ the Čech complex is defined as $\mathcal{C}_\epsilon(V)$. Its simplices are formed on every finite set of points, whose closed $\epsilon/2$-balls around the points have a non-empty intersection. Each subset of $V$ of $(k+1)$ points forms a k-simplex, which is included in the Čech complex.*

The Čech complex $\mathcal{C}$ is a subcomplex of the Vietoris-Rips complex $\mathcal{R}$, thus $\mathcal{C} \subseteq \mathcal{R}$. With the Čech complex, high dimensional simplices no longer appear, which makes the complex topologically more accurate. But still, there are also some disadvantages to the Čech complex. Its construction is very complicated, as one needs to check many $\epsilon/2$-ball intersections in order to build the complex. Even though the Vietoris-Rips complex has more simplices, it is less expensive to compute and store. This is due to the fact that it is a *flag complex*. These are abstract simplicial complexes of an abstract graph $G = (V, E)$, where its simplices are all subsets $S \subseteq V$, so that the subgraph $G[S]$ is a complete graph. In other words this means a Vietoris-Rips complex is wholly determined by its vertices and edges, so that only the 1-skeleton needs to be stored, with the rest of the complex being rebuilt later.
These different aspects of the complexes lead to the conclusion that for this thesis, the Vietoris-Rips complex is of more importance, because its advantages in computability are prevailing.

## 3.3 Applications in Sensor Networks

The applications for Vietoris-Rips complexes in real life are manifold. In general a real system is going to be some sort of network of nodes, which provide readings, represented by the data clouds. This was already briefly explained in the beginning of this section. In this subsection a concrete application will be described, which fits the use for a sensor network using the Vietoris-Rips complex and algebraic topology.

Imagine a large area covered by a forest, which is prone to forest fires. As the area is very large, it is almost impossible to be constantly monitored manually. Helicopters or planes overflying the area are probably very cost-intensive, which means this can not be done at all times. A practicable way to quickly detect fire is by installing several devices in the forest, which are able to detect the fire by smoke, heat or something similar. The problem occurring with this method is that it is going to be a hard task to install the devices in a way, so that the whole forest is covered and no small areas are out of reach of the detectors. If the forest area would be rectangular or even quadratic, the problem would be very simple, but more often than not the forests are of irregular shape with more than four corners or roundings. In order to cover such an irregular area one needs a way to find holes in the sensor network.

This problem will now be approached using algebraic topology. With the aid of the Vietoris-Rips complexes described in the previous section, the sensor network can be translated to the field of topology. The fire detecting devices are nodes of the complex, while $\epsilon$ is going to be an approximation of the range of each fire detector, within which fire can be detected. If holes appear in the complex, the Betti numbers should be used to detect them in the homology groups. There are two possible way to compute the Vietoris-Rips complex.

Firstly, each node could be equipped with a CPU and memory, so that all of them are able to calculate the complex based on the information it receives from its neighbours. It can be safely assumed that the data link range of the nodes is large enough to communicate with all other nodes within the entire area. This way, the nodes can share information about their position (ideally using GPS coordinates) and their range (which is identical for all nodes in an ideal case). Now each node can calculate the Vietoris-Rips complex based on this information.
The second method involves an additional central unit, which maintains a connection to all of the nodes in the network. This main unit is responsible for gathering the topological data and computing the complex. One advantage of this method is that the calculation is centralized, thus eliminating the need for all the other nodes to calculate the complex. Yet this can also be seen as a disadvantage, as the centralization leads to dependence on the central node. If it were to fail, the whole sensor network would stop working, which means there needs to be a certain amount of redundancy when implementing the central unit. Using a central unit is going to be the better choice, the more nodes exist in the network. Communication between all nodes is going to rise exponentially, when they have to communicate with each other, as the addition of one node adds multiple new communication links. With a central node, only one more bidirectional link is added.

In general both methods do have one big disadvantage. Every time a fire detector node malfunctions, it has to be repaired or replaced before the entire system is on-line again. This means that there is always an area which is not fully covered at this time. Quickly repairing or replacing the node manually is both expensive and impracticable. It would be a great improvement if the nodes were moveable and able to close out the hole by themselves. Therefore using unmanned aerial vehicles (UAV) equipped with fire detectors might be more expensive in the short term, but in the long term they will pay off due to the addition of more added security, as the system is less likely to be operated with holes for longer periods of time.

The idea of using UAVs to quickly cover holes works in a simple principle. As the central node constantly determines the Betti numbers for the topological complex, a hole can be detected very quickly. If unused UAVs were available at this time, they would be deployed to the exact same spot the now defunct one is at. If no unused UAVs are available, it is the task of the central unit to arrange the remaining UAVs in a new pattern, so that the maximal area of the forest is encircled by the range of the detectors. As computing every possible new Vietoris-Rips complex is very expensive in terms of computing power, it is practicable to just use an UAV from the outer edges of the forest and deploy it, so that the hole is covered again. Even though this is going to eliminate one or two edges on the outside of the complex, the much more critical hole problem is solved. If the defunct UAV is repaired or replaced by a new one, the old arrangement can be used again.

The sensor network system described above guarantees a fire detection coverage of the forest at most times, with the exception of only small periods, when a malfunction occurred. These periods can be further reduced if redundancy is added to the sensor network. Especially in areas, which are particularly prone to fire, it might be practical to deploy more UAVs than necessary. Failing nodes would now affect the network much less than before, as at least a large part of the hole is still covered due to the added UAVs.
Another possibility for redundancy would be an increase of the range of the detectors (represented by $\epsilon$ in the Vietoris-Rips complex) beyond the minimal range required for the nodes to form simplices of a certain dimension. This would of course result in more higher dimensional simplices (at least in the high risk area) and therefore a more complicated complex representation.
Which of the methods is preferable strongly depends on several side aspects. Adding more UAVs to the network is expensive, but might be less complicated for the topological representation. Whereas increasing the detection range is only possible to a limited extent without having to acquire newer and more powerful fire detection devices. Both methods have their disadvantages, but this is a price, which one might be willing to pay for the added safety.

One more aspect of the application needs to be discussed here. So far, this chapter was mainly focused an a 2-dimensional representation of the sensor network. In reality, for most forests the third dimension can be safely omitted, as the height differences are insignificant compared to the other measurements. But still, especially in a mountainous region the Vietoris-Rips complex would have to be used in $\mathbb{R}^3$ in order to be accurate. Technically, it would work in a very similar way. Instead of circles of diameter $\epsilon$ around the points, now spheres have to be used. This leads to the whole calculation of pairs, triangles and higher dimensional simplices becoming more complicated due to the added dimension.

# 4 Conclusion and Outlook

One possible application of sensor networks using algebraic topology in detecting forest fires has been described in this thesis, but there are several other possibilities in which the algebraic topology can be used in sensor networks. There is for example the mountain rescue scenario, in which a mountain, or a mountain range is monitored by sensors, preferably also UAVs. These sensors would be searching lost people in the area and automatically forward their position the mountain rescue service.
Another application could be in the field of earthquake prediction. Sensor nodes could detect seismic activity in the ground. Of course in this application using UAVs is not an option and thus it would be more complicated to change the physical location of the detectors and with that the underlying topological representation in the abstract simplicial complex. But the idea still remains applicable for that scenario.

There is a lot of improvement that can be done by future research in this area. Especially the autonomous coordination and communication of the UAVs (with or without a central unit) is a task, which is rather difficult to realise. Also using the Čech complex instead of the Vietoris-Rips complex can improve the accuracy of the topological representation. Even though there are the disadvantages that have been mentioned, it should be possible to use the complex in a way its advantages outweigh them.

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