Intelligent Autonomous Agents

Agents and Rational Behavior: Uncertainty

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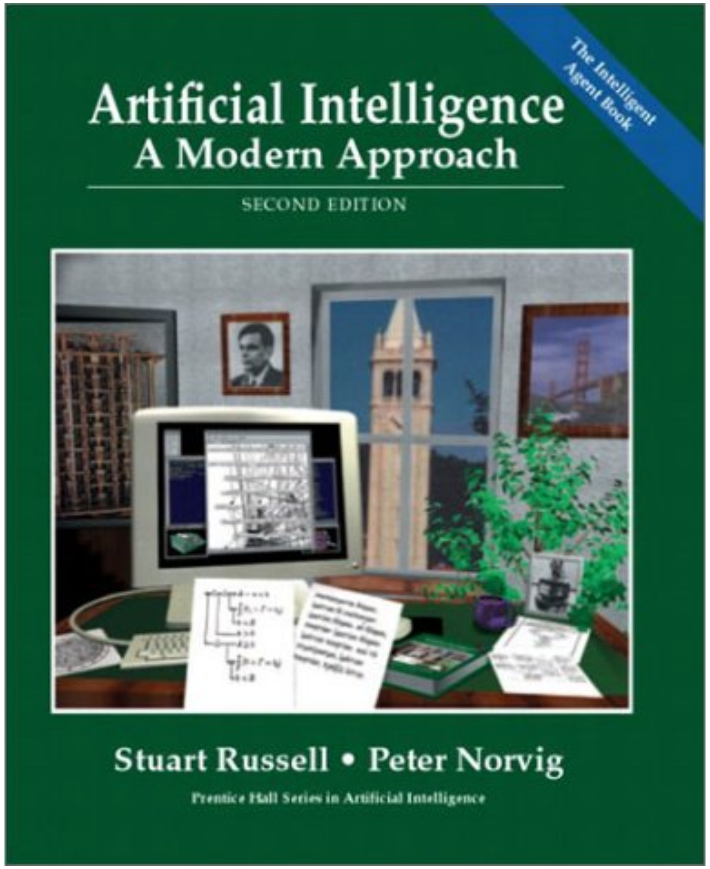
Literature

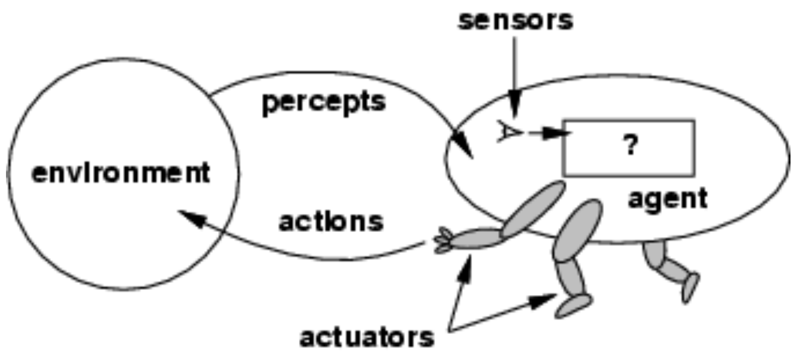


- Chapter 13

Outline

- Agents
- Uncertainty
- Probability
- Syntax and Semantics
- Inference
- Independence and Bayes' Rule
- Bayesian Networks

Recap: Agents and environments



- The agent function maps from percept histories to actions:

$$[f: \mathcal{P}^* \rightarrow \mathcal{A}]$$

- The agent program runs on the physical architecture to produce f

- agent = architecture + program

Uncertainty

Let action A_t = leave for airport t minutes before flight
Will A_t get me there on time?

Problems:

1. partial observability (road state, other drivers' plans, etc.)
2. noisy sensors (traffic reports)
3. uncertainty in action outcomes (flat tire, etc.)
4. immense complexity of modeling and predicting traffic

Hence, it seems that a simple standard logical approach either

1. risks falsehood: "A_{25} will get me there on time", or
2. leads to conclusions that are too weak for decision making:

"A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact etc etc."

(A_{1440} might reasonably be said to get me there on time but I'd have to stay overnight in the airport …)

Methods for handling uncertainty

- Logic:
  - Assume my car does not have a flat tire
  - Assume A_{25} works unless contradicted by evidence
  - Issues:
    - Which assumptions are reasonable?
    - How to handle contradiction?
    - Possible but theory of preferred models is hard.
- Rules with fudge factors (belief in the rule):
  - $A_{25} \mid\!\rightarrow_{0.3}$ get there on time
  - $Sprinkler \mid\!\rightarrow_{0.99} WetGrass$
  - $WetGrass \mid\!\rightarrow_{0.7} Rain$
  - Issues:
    - Problem with semantics
    - Problems with combination, e.g., *Sprinkler* causes *Rain*??

Handling uncertainty (cntd.)

- Propositional Logic and Probability Theory
  - Model agent's degree of belief
  - Given the available evidence,
  - A_{25} will get me there on time with probability 0.00001
- Predicate Logic and Probability Theory
  - Can additionally talk about possibly existing objects and possible relations
  - Stay tuned for future lectures …

Probability

Probabilistic assertions summarize effects of

- Laziness: failure to enumerate exceptions, qualifications, etc.
- Ignorance: lack of relevant facts, initial conditions, etc.

Subjective probability:

- Probabilities relate propositions to agent's own state of knowledge, e.g.,

  $P(A_{25} \mid \text{no reported accidents}) = 0.06$

These are not assertions about the world

Probabilities of propositions change with new evidence, e.g.,

$P(A_{25} \mid \text{no reported accidents, 5 a.m.}) = 0.15$

Making decisions under uncertainty

Suppose I believe the following:

$P(A_{25}$ gets me there on time | …) = 0.04

$P(A_{90}$ gets me there on time | …) = 0.70

$P(A_{120}$ gets me there on time | …) = 0.95

$P(A_{1440}$ gets me there on time | …) = 0.9999

- Which action to choose?

  Depends on my preferences for missing flight vs. time spent waiting, etc.

  - Utility theory is used to represent and infer preferences
  - Decision theory = probability theory + utility theory

Probability theory: syntax

- Basic element: random variable
- Similar to propositional logic: possible worlds defined by assignment of values to random variables.
- Boolean random variables
  e.g., *Cavity* (do I have a cavity?). Domain is <true, false>
- Discrete random variables
  e.g., *Weather* is one of <*sunny,rainy,cloudy,snow*>
- Domain values must be exhaustive and mutually exclusive
- Elementary proposition constructed by assignment of a value to a random variable: e.g.,
  - *Weather = sunny*,
  - *Cavity = false* (abbreviated as $\neg$*cavity*)
  - *Cavity = true* (abbreviated as *cavity*)
- Complex propositions formed from elementary propositions and standard logical connectives e.g., *Weather = sunny* $\vee$ *Cavity = false*

Syntax

- Atomic event: A complete specification of the state of the world about which the agent is uncertain

  E.g., if the world is described by only two Boolean variables *Cavity* and *Toothache*, then there are 4 distinct atomic events:

  $Cavity = false \wedge Toothache = false$

  $Cavity = false \wedge Toothache = true$

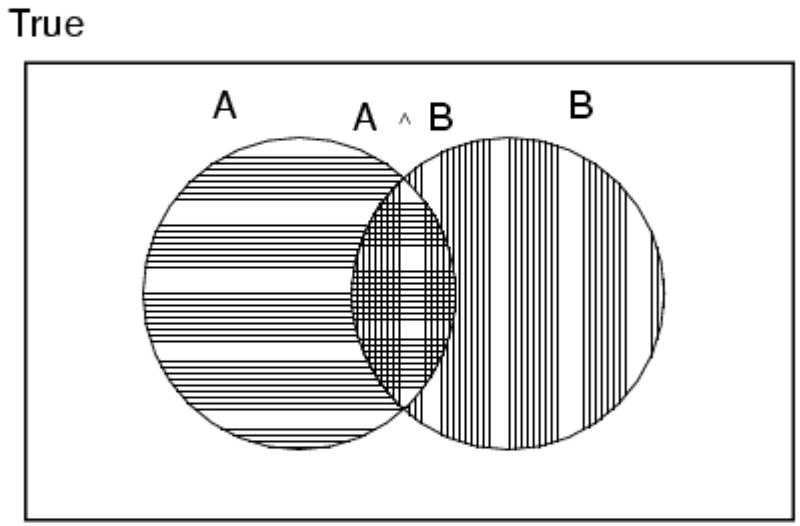
  $Cavity = true \wedge Toothache = false$

  $Cavity = true \wedge Toothache = true$

- Atomic events are mutually exclusive and exhaustive

Axioms of probability

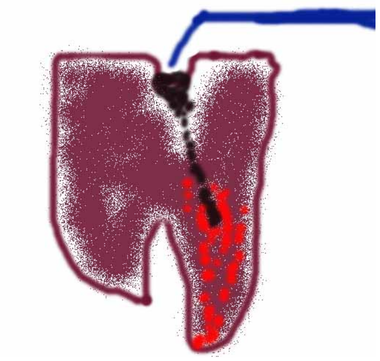
- For any propositions *A*, *B*
  - $0 \leq P(A) \leq 1$
  - $P(true) = 1$ and $P(false) = 0$
  - $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$



Example world

Example: *Dentist problem* with four variables:

- *Toothache* (I have a toothache)
- *Cavity* (I have a cavity)
- *Catch* (steel probe catches in my tooth)
- *Weather* (*sunny,rainy,cloudy,snow*)



Prior probability

- Prior or unconditional probabilities of propositions

  e.g., P(*Cavity* = true) = 0.1 and P(*Weather* = sunny) = 0.72
  correspond to belief prior to arrival of any (new) evidence

- Probability distribution
  gives values for all possible assignments:

  P(*Weather*) = <0.72,0.1,0.08,0.1>
  (normalized, i.e., sums to 1)

Full joint probability distribution

- Joint probability distribution for a set of random variables gives the probability of every atomic event on those random variables

P(*Weather*,*Cavity*) = a 4 × 2 matrix of values:

Weather =	sunny	rainy	cloudy	snow
Cavity = true	0.144	0.02	0.016	0.02
Cavity = false	0.576	0.08	0.064	0.08

- Full joint probability distribution: all random variables involved
  - **P**(Toothache, Catch, Cavity, Weather)
- Every question about a domain can be answered by the full joint distribution

Conditional probability

- Conditional or posterior probabilities
  e.g., P(*cavity* | *toothache*) = 0.8
  or: <0.8>
  i.e., given that *toothache* is all I know

- (Notation for conditional distributions:
  P(*Cavity* | *Toothache*) = 2-element vector of 2-element vectors)

- If we know more, e.g., *cavity* is also given, then we have
  P(*cavity* | *toothache,cavity*) = 1

- New evidence may be irrelevant, allowing simplification, e.g.,
  P(*cavity* | *toothache, sunny*) = P(*cavity* | *toothache*) = 0.8

- This kind of inference, sanctioned by domain knowledge, is crucial

Conditional probability

- Definition of conditional probability (in terms of uncond. prob.):
  $P(a \mid b) = P(a \wedge b) / P(b)$ if $P(b) > 0$

- Product rule gives an alternative formulation ($\wedge$ is commutative):
  $P(a \wedge b) = P(a \mid b)\, P(b) = P(b \mid a)\, P(a)$

- A general version holds for whole distributions, e.g.,
  $\mathbf{P}(Weather, Cavity) = \mathbf{P}(Weather \mid Cavity)\, \mathbf{P}(Cavity)$
  View as a set of 4×2 equations, not matrix mult.
  (1,1) $P(Weather=sunny \mid Cavity=true)\, P(Cavity=true)$
  (1,2) $P(Weather=sunny \mid Cavity=false)\, P(Cavity=false)$, ….

- Chain rule is derived by successive application of product rule:

$$\begin{aligned}
\mathbf{P}(X_1, \ldots, X_n) &= \mathbf{P}(X_1,\ldots,X_{n-1})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1}) \\
&= \mathbf{P}(X_1,\ldots,X_{n-2})\, \mathbf{P}(X_{n-1} \mid X_1,\ldots,X_{n-2})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1}) \\
&= \ldots \\
&= \prod_{i=1}^{n} \mathbf{P}(X_i \mid X_1, \ldots, X_{i-1})
\end{aligned}$$

Inference by enumeration

- Start with the joint probability distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- For any proposition φ, sum the atomic events where it is true: $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$

Inference by enumeration

- Start with the joint probability distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

- For any proposition φ, sum the atomic events where it is true: $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$
- P(*toothache*) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2
- Unconditional or **marginal probability** of toothache
- Process is called marginalization or summing out

Marginalization and conditioning

- Let Y, Z be sequences of random variables s.th. Y \cup Z denotes all random variables describing the world

- Marginalization
  - $\mathbf{P}(\mathbf{Y}) = \Sigma_{\mathbf{z} \text{ in } \mathbf{Z}} \mathbf{P}(\mathbf{Y},\mathbf{z})$

- Conditioning
  - $\mathbf{P}(\mathbf{Y}) = \Sigma_{\mathbf{z} \text{ in } \mathbf{Z}} \mathbf{P}(\mathbf{Y}|\mathbf{z})P(\mathbf{z})$

Inference by enumeration

- Start with the joint probability distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

For any proposition φ, sum the atomic events where it is true:
$P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$

- P(cavity ∨ *toothache*) = 0.108 + 0.012 + 0.072 + 0.008+ 0.016 + 0.064 = 0.28

(P(*cavity* ∨ *toothache*) = P(*cavity*) + P(*toothache*) – P(*cavity* ∧ *toothache*))

Inference by enumeration

- Start with the joint probability distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

- Can also compute conditional probabilities:

$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \land toothache)}{P(toothache)}$$

Product rule

$$= \frac{0.016+0.064}{0.108 + 0.012 + 0.016 + 0.064}$$

$$= 0.4$$

Normalization

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

- Denominator **P**(**z**) (or P(toothache) in the example before) can be viewed as a normalization constant α

$$\mathbf{P}(Cavity \mid toothache) = \alpha\, \mathbf{P}(Cavity, toothache)$$
$$= \alpha\, [\mathbf{P}(Cavity, toothache, catch) + \mathbf{P}(Cavity, toothache, \neg\, catch)]$$
$$= \alpha\, [\langle 0.108, 0.016\rangle + \langle 0.012, 0.064\rangle]$$
$$= \alpha\, \langle 0.12, 0.08\rangle = \langle 0.6, 0.4\rangle$$

General idea: compute distribution on query variable by fixing evidence variables (Toothache) and summing over hidden variables (Catch)

Inference by enumeration, contd.

Typically, we are interested in
the posterior joint distribution of the query variables **Y**
given specific values **e** for the evidence variables **E**

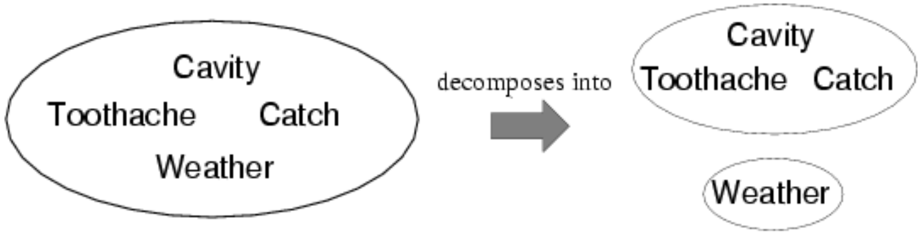
Let the hidden variables be **H** = **X** - **Y** - **E** then the required summation of joint entries is done by summing out the hidden variables:

$$\mathbf{P}(\mathbf{Y} \mid \mathbf{E} = \mathbf{e}) = \alpha \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}) = \alpha \Sigma_{h} \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}, \mathbf{H} = \mathbf{h})$$

- The terms in the summation are joint entries because **Y**, **E** and **H** together exhaust the set of random variables
- Obvious problems:
  1. Worst-case time complexity $O(d^n)$ where d is the largest arity and n denotes the number of random variables
  2. Space complexity $O(d^n)$ to store the joint distribution
  3. How to find the numbers for $O(d^n)$ entries?

Independence

- A and B are independent iff
  $\mathbf{P}(A|B) = \mathbf{P}(A)$ or $\mathbf{P}(B|A) = \mathbf{P}(B)$ or $\mathbf{P}(A, B) = \mathbf{P}(A)\,\mathbf{P}(B)$



$\mathbf{P}(Toothache, Catch, Cavity, Weather)$
$= \mathbf{P}(Toothache, Catch, Cavity)\,\mathbf{P}(Weather)$

- 32 entries reduced to 12;
- for n independent biased coins, $O(2^n) \rightarrow O(n)$
- Absolute independence powerful but rare
- Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

Conditional independence

- **P**(*Toothache, Cavity, Catch*) has $2^3 - 1 = 7$ independent entries
- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

  (1) **P**(*catch* | *toothache, cavity*) = **P**(*catch* | *cavity*)

- The same independence holds if I haven't got a cavity:

  (2) **P**(*catch* | *toothache*,¬*cavity*) = **P**(*catch* | ¬*cavity*)

- *Catch* is conditionally independent of *Toothache* given *Cavity*:

  P(*Catch* | *Toothache,Cavity*) = **P**(*Catch* | *Cavity*)

- Equivalent statements:

  P(*Toothache* | *Catch, Cavity*) = **P**(*Toothache* | *Cavity*)

  P(*Toothache, Catch* | *Cavity*) = **P**(*Toothache* | *Cavity*) **P**(*Catch* | *Cavity*)

Conditional independence contd.

- Write out full joint distribution using chain rule:

  $\mathbf{P}(Toothache, Catch, Cavity)$

  $= \mathbf{P}(Toothache \mid Catch, Cavity)\ \mathbf{P}(Catch, Cavity)$

  $= \mathbf{P}(Toothache \mid Catch, Cavity)\ \mathbf{P}(Catch \mid Cavity)\ \mathbf{P}(Cavity)$

  $= \mathbf{P}(Toothache \mid Cavity)\ \mathbf{P}(Catch \mid Cavity)\ \mathbf{P}(\text{Cavity})$

  I.e., 2 + 2 + 1 = 5 independent numbers

- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in *n* to linear in *n*.

Bayes' Rule

Product rule $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

$\Rightarrow$ Bayes' rule $P(a|b) = \dfrac{P(b|a)P(a)}{P(b)}$

or in distribution form

$$\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha \mathbf{P}(X|Y)\mathbf{P}(Y)$$

Useful for assessing diagnostic probability from causal probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

E.g., let M be meningitis, S be stiff neck:

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

Note: posterior probability of meningitis still very small!

Bayes' Rule (2)

$$\mathbf{P}(Cavity|toothache \wedge catch)$$
$$= \alpha\, \mathbf{P}(toothache \wedge catch|Cavity)\mathbf{P}(Cavity)$$
$$= \alpha\, \mathbf{P}(toothache|Cavity)\mathbf{P}(catch|Cavity)\mathbf{P}(Cavity)$$

This is an example of a naive Bayes model:

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause)\Pi_i \mathbf{P}(Effect_i|Cause)$$

Total number of parameters is linear in n