Multimedia Information Extraction and Retrieval

Latent Semantic Analysis

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Acknowledgements

- Slides taken from presentation material for the following book:

Introduction to Information Retrieval

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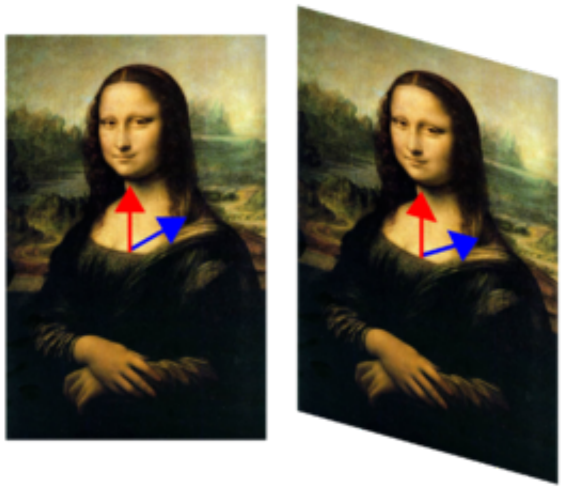
Mapping Data

$$\mathbf{S}\mathbf{v} = \lambda\mathbf{v}$$



http://de.wikipedia.org/wiki/Eigenvektor

Eigenvalues & Eigenvectors

- **Eigenvectors** (for a square $m \times m$ matrix $\mathbf{S}$)

$$\mathbf{S}\mathbf{v} = \lambda \mathbf{v}$$

(right) eigenvector $\mathbf{v} \in \mathbb{R}^m \neq \mathbf{0}$

eigenvalue $\lambda \in \mathbb{R}$

Example

$$\begin{pmatrix} 6 & -2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- How many eigenvalues are there at most?

$$\mathbf{S}\mathbf{v} = \lambda \mathbf{v} \iff (\mathbf{S} - \lambda \mathbf{I})\,\mathbf{v} = \mathbf{0}$$

only has a non-zero solution if $|\mathbf{S} - \lambda \mathbf{I}| = 0$

this is a m-th order equation in λ which can have ***at most m distinct solutions*** *(roots of the characteristic polynomial) - can be complex even though **S** is real.*

Singular Value Decomposition

For an m× n matrix $\mathbf{A}$ *of rank* r *there exists a factorization (Singular Value Decomposition =* ***SVD****) as follows:*

$$A = U\Sigma V^T$$

m×m | *m×n* | *V is n×n*

The columns of ***U*** *are orthogonal eigenvectors of* $\mathbf{AA^T}$.

The columns of ***V*** *are orthogonal eigenvectors of* $\mathbf{A^TA}$.

Eigenvalues $\lambda_1 \ldots \lambda_r$ *of* $\mathbf{AA^T}$ *are the eigenvalues of* $\mathbf{A^TA}$.

$$\sigma_i = \sqrt{\lambda_i}$$

$$\Sigma = diag(\sigma_1 \ldots \sigma_r)$$

Singular values.

SVD example

Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Thus m=3, n=2. Its SVD is

$$\begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

As opposed to the presentation in the example, typically, the singular values arranged in decreasing order.

Low-rank Approximation

- SVD can be used to compute optimal **low-rank approximations**.
- Approximation problem: Find A_k of rank k such that

$$A_k = \underset{X:rank(X)=k}{\arg\min} \|A - X\|_F$$

← *Frobenius norm*

$$\|A\|_F \equiv \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}.$$

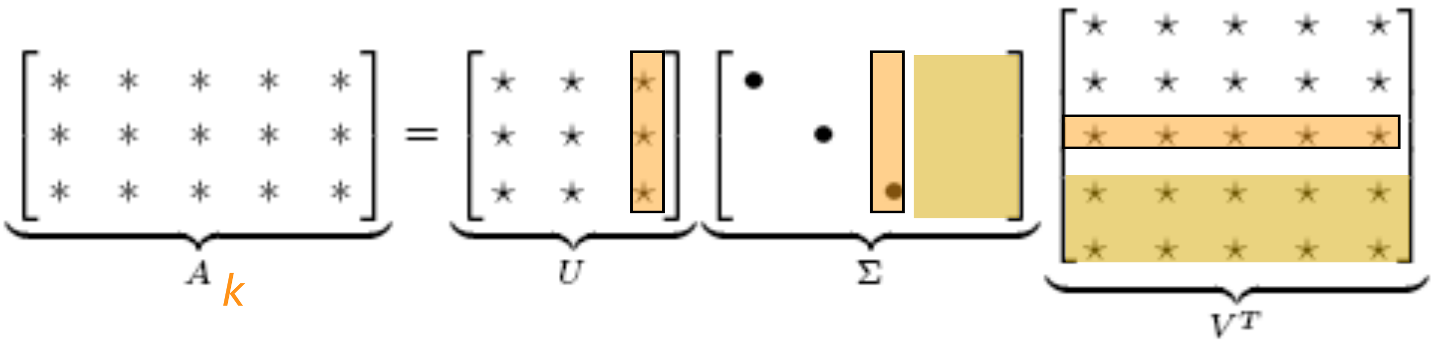
A_k and X are both $m \times n$ matrices.

Typically, want $k << r$.

Low-rank Approximation

- Solution via SVD

$$A_k = U \operatorname{diag}(\sigma_1, \ldots, \sigma_k, \underbrace{0, \ldots, 0}_{\text{set smallest r-k singular values to zero}}) V^T$$



SVD Low-rank approximation

- Whereas the term-doc matrix *A* may have *m*=50000, *n*=10 million (and rank close to 50000)
- We can construct an approximation A_{100} with rank 100.
  - Of all rank 100 matrices, it would have the lowest Frobenius error.
- Great … but why would we??
- Answer: *Latent Semantic Indexing*

C. Eckart, G. Young, The approximation of a matrix by another of lower rank. Psychometrika, 1, 211-218, 1936.

What it is

- From term-doc matrix A, we compute the approximation $A_{k.}$
- There is a row for each term and a column for each doc in A_k
- Thus docs live in a space of $k \ll r$ dimensions
  - These dimensions are not the original axes
- But why?

Vector Space Model: Pros

- **Automatic** selection of index terms
- **Partial matching** of queries and documents *(dealing with the case where no document contains all search terms)*
- **Ranking** according to **similarity score** *(dealing with large result sets)*
- **Term weighting** schemes *(improves retrieval performance)*
- Geometric foundation

Problems with Lexical Semantics

- Ambiguity and association in natural language
  - **Polysemy**: Words often have a **multitude of meanings** and different types of usage *(more severe in very heterogeneous collections).*
  - The vector space model is unable to discriminate between different meanings of the same word.

$$\mathrm{sim}_{\mathrm{true}}(d, q) < \cos(\angle(\vec{d}, \vec{q}))$$

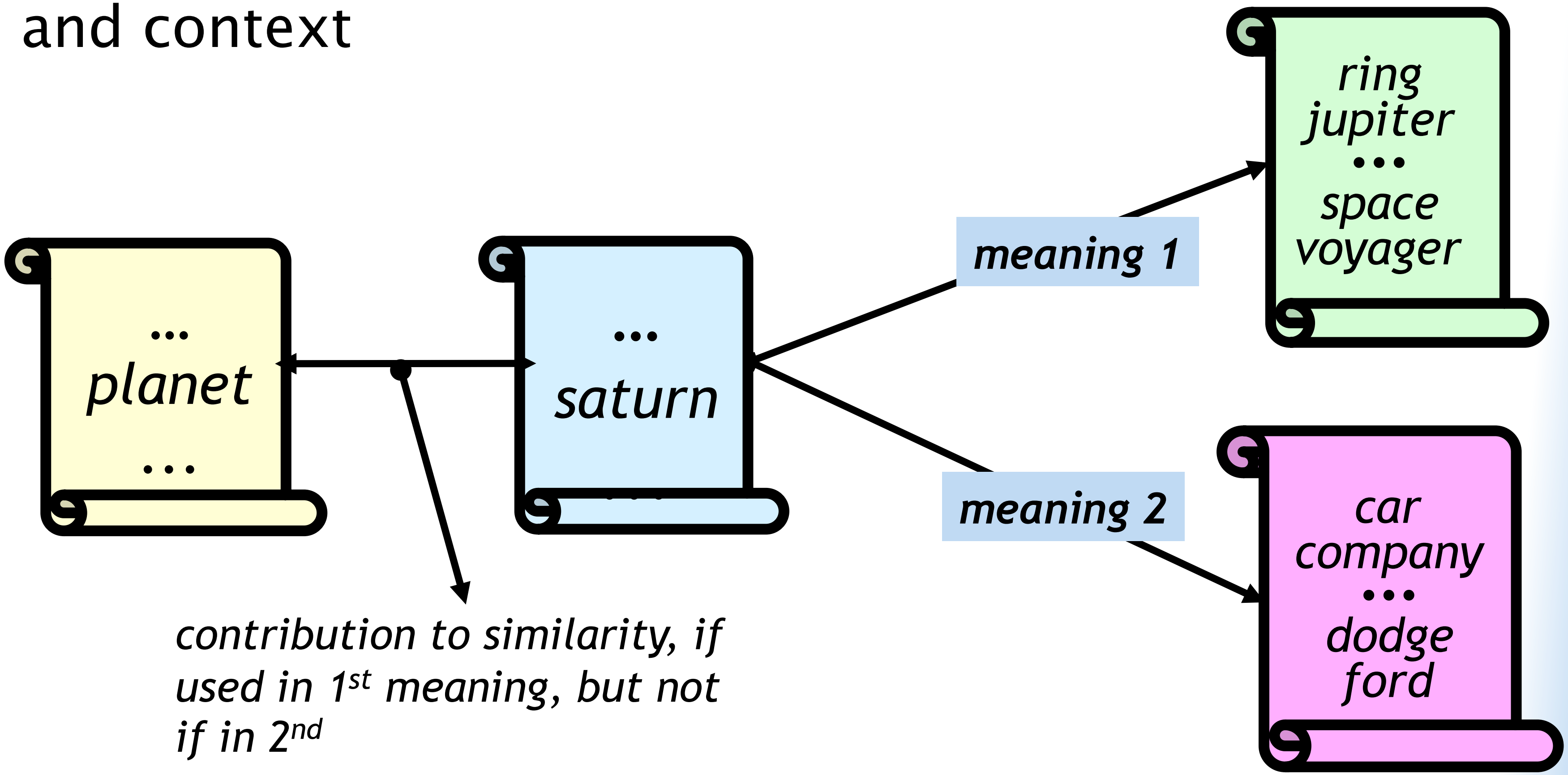
Problems with Lexical Semantics

- **Synonymy**: Different terms may have an **dentical or a similar meaning** (weaker: words indicating the same topic).
- No associations between words are made in the vector space representation.

$$\mathrm{sim}_{\mathrm{true}}(d, q) > \cos(\angle(\vec{d}, \vec{q}))$$

Polysemy and Context

- Document similarity on single word level: polysemy and context



Latent Semantic Indexing (LSI)

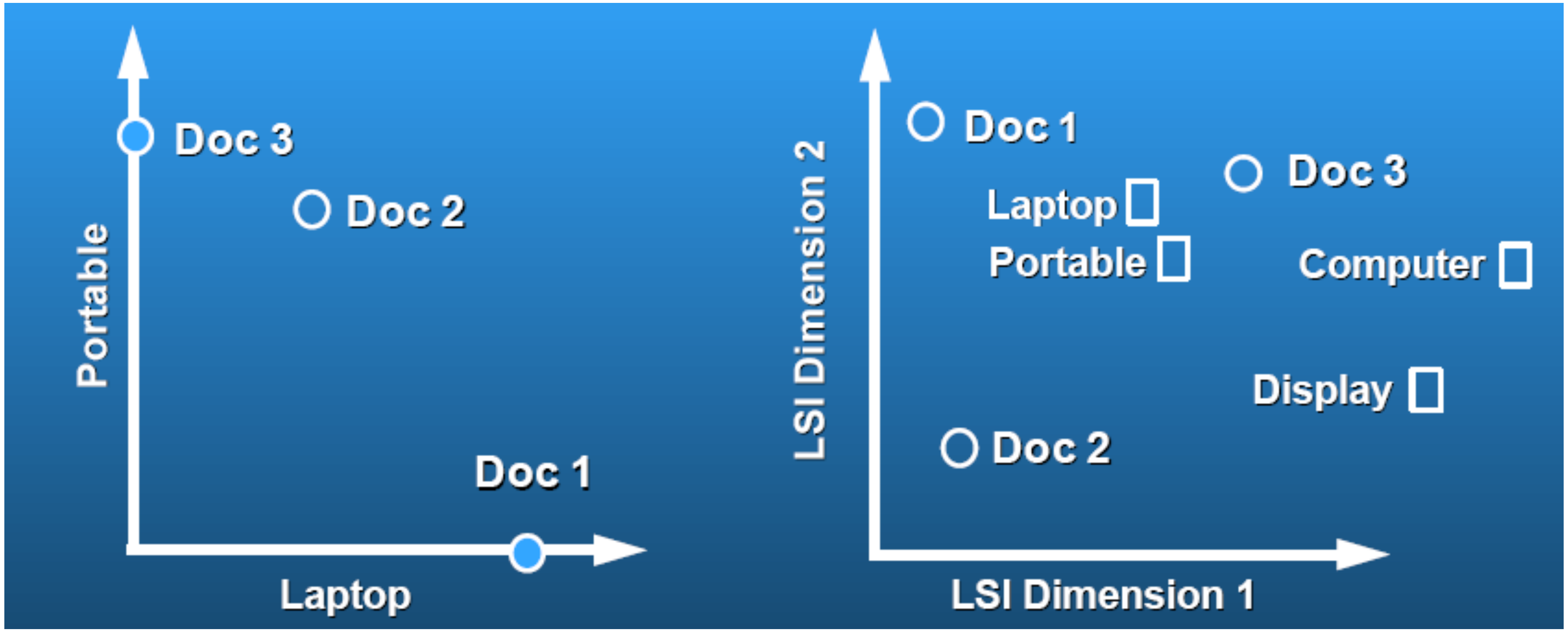
- Perform a **low-rank approximation** of **document-term matrix** (typical rank **100-300**)
- General idea
  - Map documents (*and* terms) to a **low-dimensional** representation.
  - Design a mapping such that the low-dimensional space reflects **semantic associations** (latent semantic space).
  - Compute document similarity based on the **inner product** in this **latent semantic space**

Goals of LSI

- Similar terms map to similar location in low dimensional space
- Noise reduction by dimension reduction

Latent Semantic Analysis

- **Latent semantic space**: illustrating example



courtesy of Susan Dumais

Performing the maps

- Each row and column of *A* gets mapped into the *k*–dimensional LSI space, by the SVD.
- Claim – this is not only the mapping with the best (Frobenius error) approximation to *A*, but in fact *improves* retrieval.
- A query q is also mapped into this space, by

$$q_k = q^T U_k \Sigma_k^{-1}$$

  - Query NOT a sparse vector.

Empirical evidence: TREC

- Generally expect recall to improve – what about precision?
- Precision at or above median TREC precision
  - Top scorer on almost 20% of TREC topics
- Slightly better on average than straight vector spaces
- Effect of dimensionality:

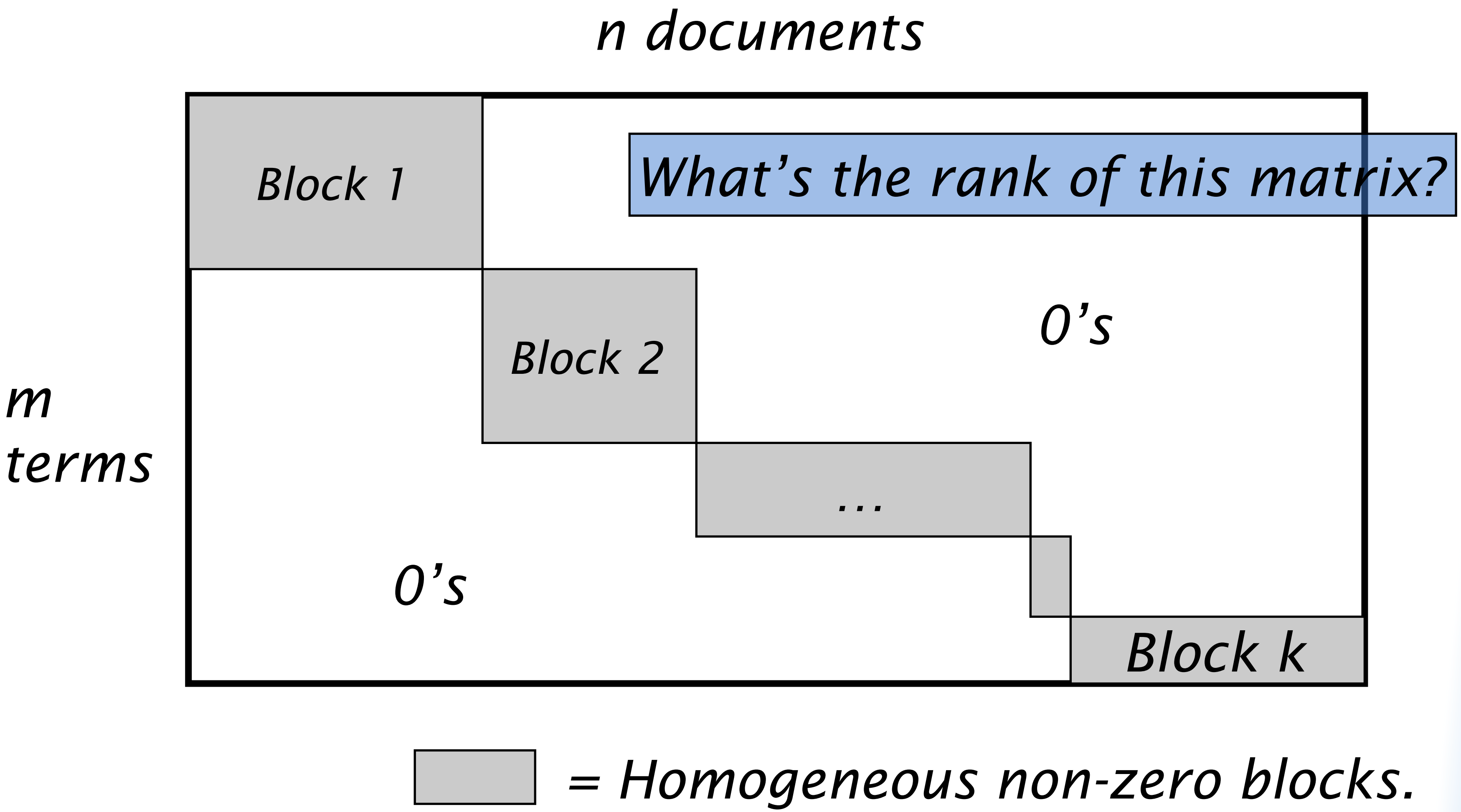
Dimensions	Precision
250	0.367
300	0.371
346	0.374

TREC = Text REtrieval Conference benchmarks

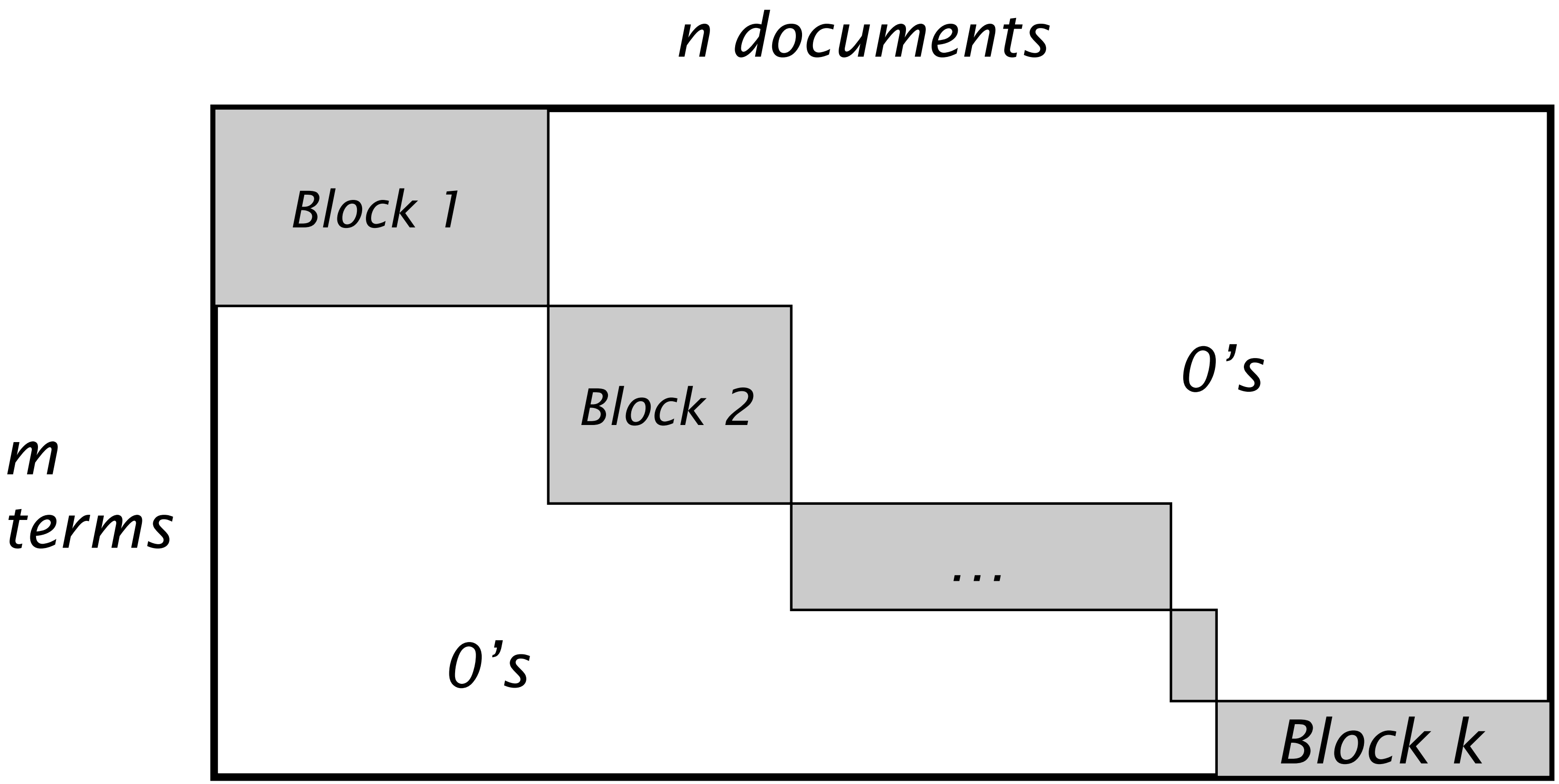
LSA seen as clustering

- We've talked about docs, queries, retrieval and precision here.
- What does this have to do with clustering?
- Intuition: Dimension reduction through LSI brings together "related" axes in the vector space.

Intuition from block matrices

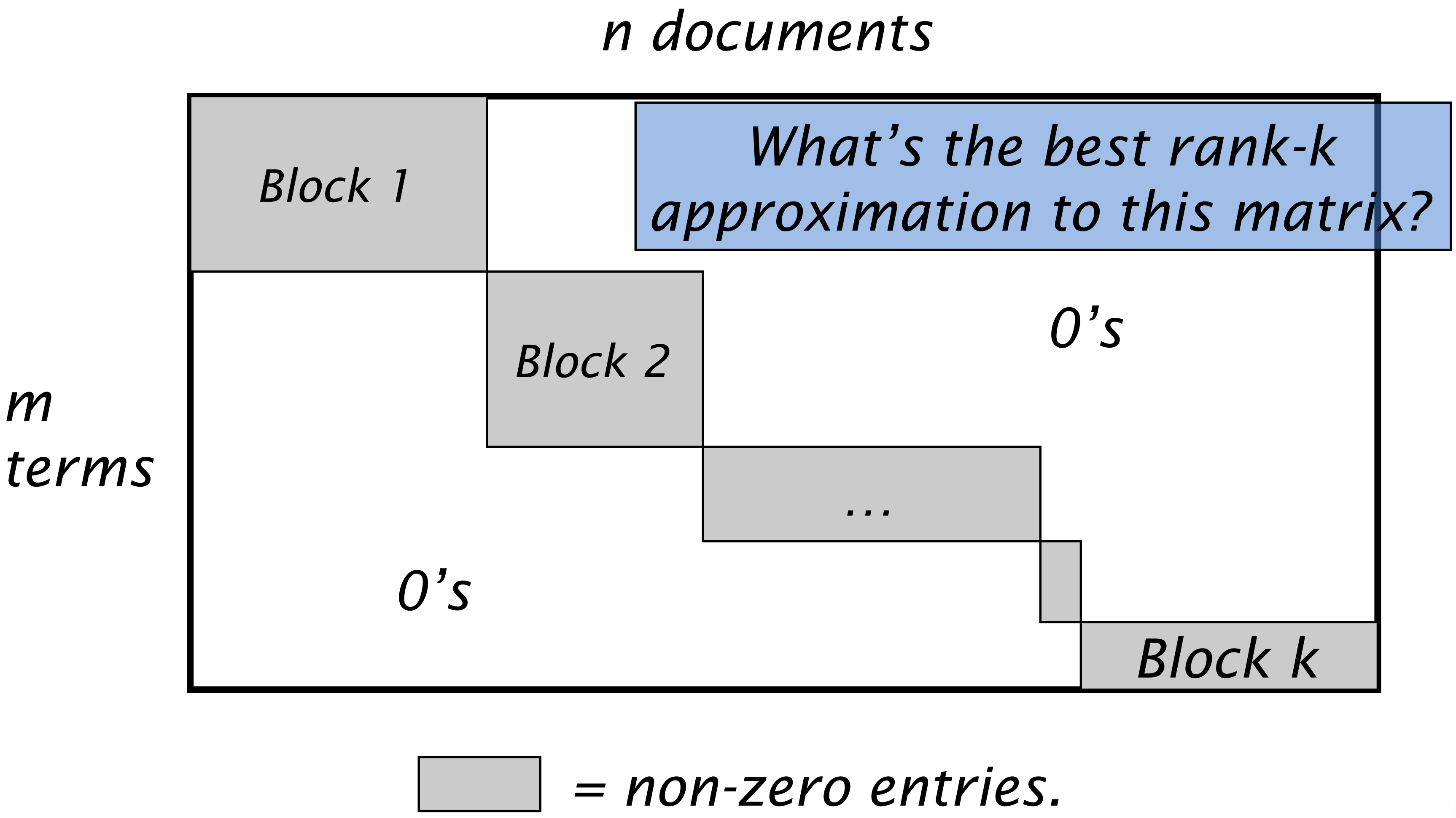


Intuition from block matrices



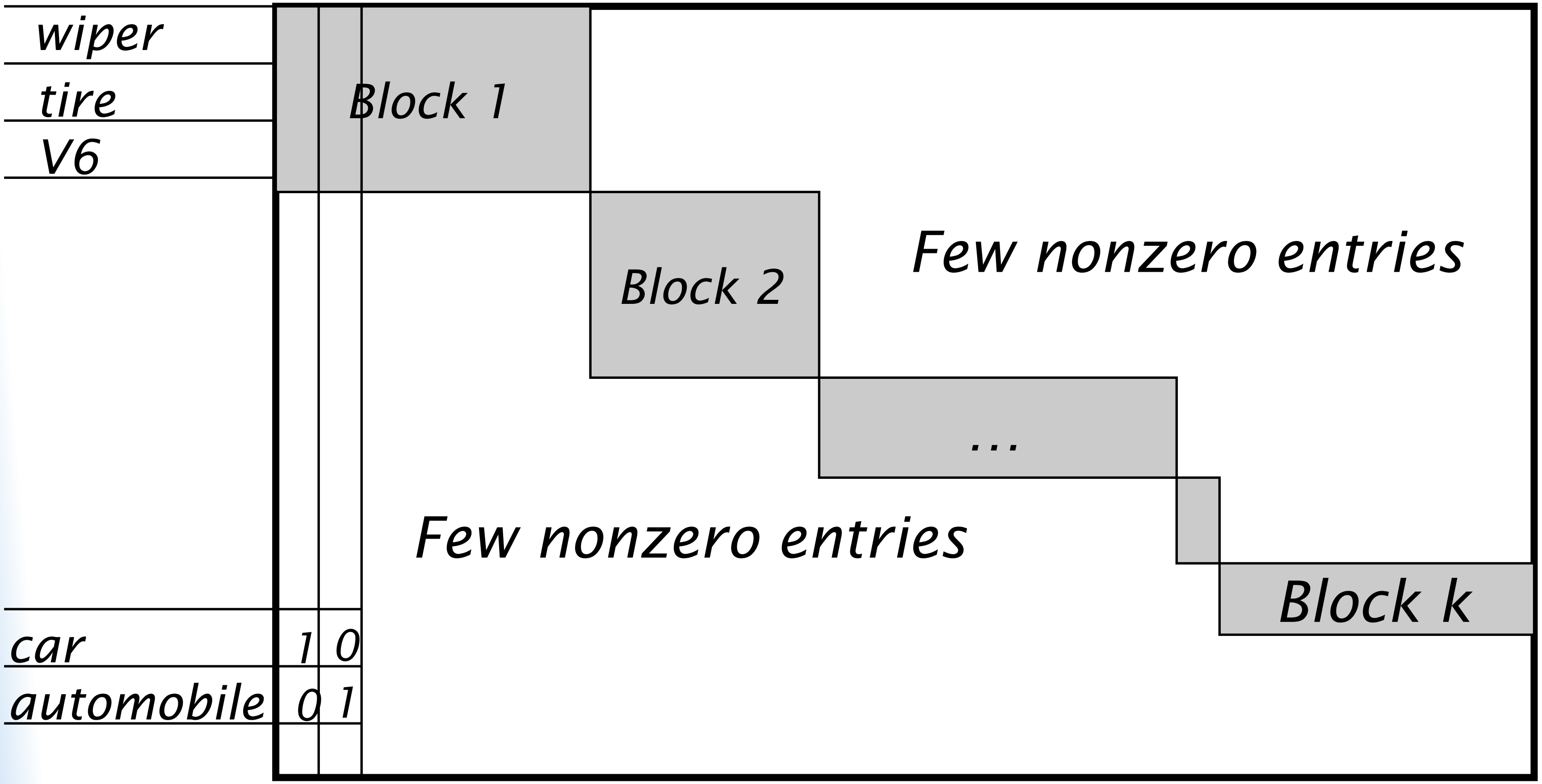
Vocabulary partitioned into k topics (clusters); each doc discusses only one topic.

Intuition from block matrices

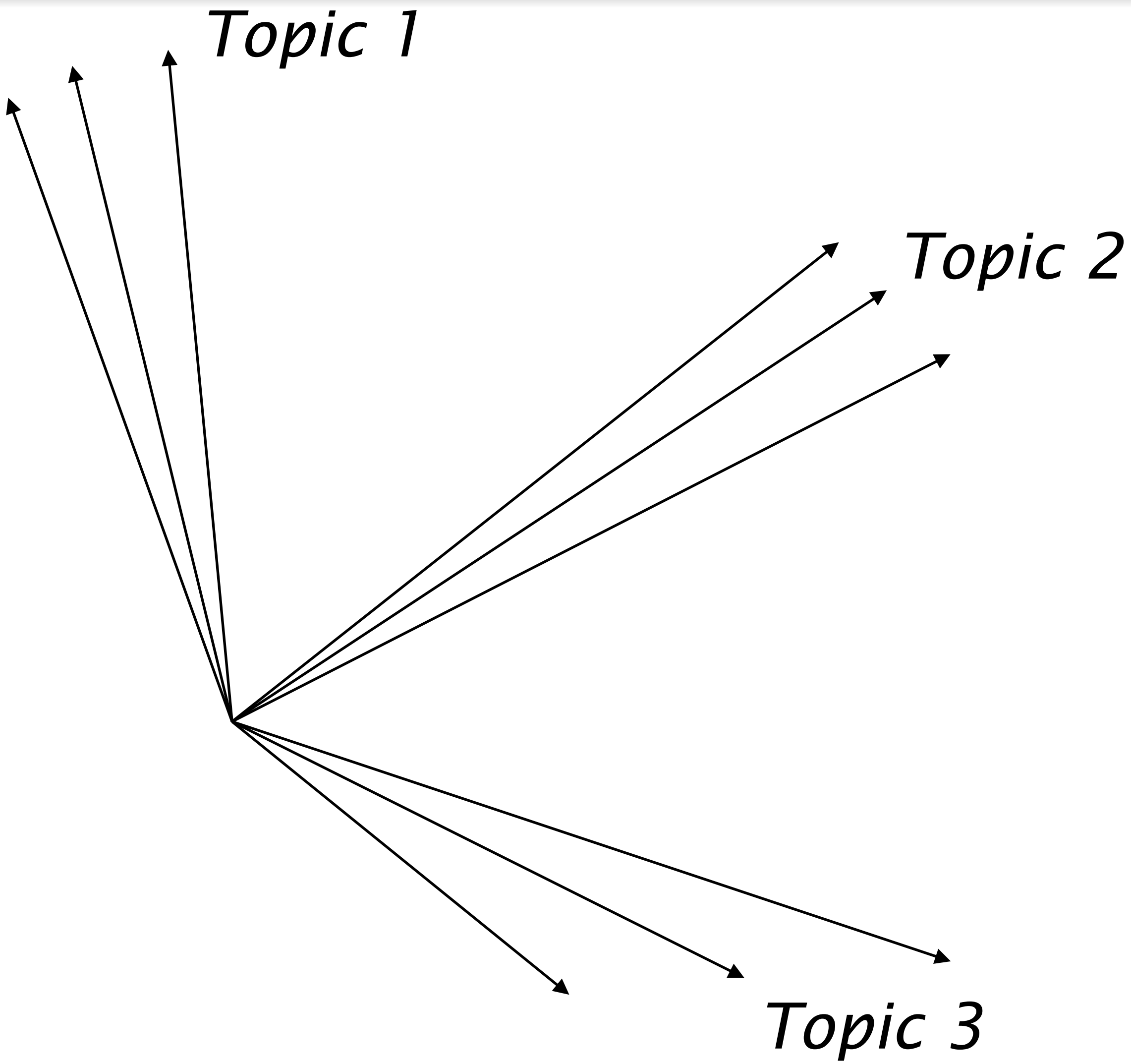


Intuition from block matrices

Likely there's a good rank-k approximation to this matrix.



Simplistic picture



Some wild extrapolation

- The "dimensionality" of a corpus is the number of distinct topics represented in it.
- More mathematical wild extrapolation:
  - if A has a rank k approximation of low Frobenius error, then there are no more than k distinct topics in the corpus.