UNIVERSITÄT ZU LÜBECK
INSTITUT FÜR INFORMATIONSSYSTEME

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Finite Model Theory

Lecture 3: Motivation, Games, Reduction Tricks
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Informationssysteme CS4130
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Recap of Lecture 2: FOL

FOL as a Representation Language

- FOL provides expressive language with neat semantics to represent assertions relevant for CS
  - System descriptions
  - Desired requirements
  - System behavior description
  - Domain constraints

Solving Algorithmic Problems in FOL

- Definitions for important semantical properties (satisfaction, satisfiability, entailment) do not tell how to compute them

- Proof calculi to the rescue
- Various FOL calculi exist that have desired properties of being correct and complete
- Prominent ones that are "directed" and hence well implementable: Tableaux and Resolution

- Resolution calculi
  - Refutation calculus (un-satisfiability tester)
  - Data structure: Formula in Clausal Normal Form
  - Resolution rule:

$$(A \vee \neg B) \wedge (B \vee C) \vDash_{res} A \vee C$$

Solving Algorithmic Problems in FOL

- No decidability for validity (unsatisfiability, entailment) but semi-decidability
- Hence we will have to consider different variants of FOL
- Undecidability still holds when changing to finite model semantics; the situation is even worse:

Theorem (Trakhtenbrot)

Validity of FOL sentences under finite model semantics is not semi-decidable

- Nonetheless FOL has important role (for CS)
  - FOL "open" (has parameters) for restrictions to more feasible fragments: number of variables, predicates, arity of predicates, complex formulae construction, quantifier nesting, quantifier alternation etc.
  - FOL (per se) is useful as a query language on DBs: constant time in data complexity ($\Longrightarrow$ to be discussed today)

End of Recap

Literature Hints

- **Lit:** L. Libkin. The finite model theory toolbox of a database theoretician. In PODS '09: Proceedings of the twenty-eighth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems, pages 65–76, New York, NY, USA, 2009. ACM.
- **Lit:** L. Libkin. Elements Of Finite Model Theory (Texts in Theoretical Computer Science. An Eatcs Series). SpringerVerlag, 2004.
- **Lit:** H. Ebbinghaus and J. Flum. Finite Model Theory. Perspectives in mathematical logic. Springer, 1999.

Aim

Understand: “Finite Model Theory (FMT) is the backbone of database theory”

Finite Model Theory

- Fundamental ideas
  1. Consider DBs as finite FOL structures
  2. Consider FOL as query language over DBs
- Starting with FOL investigate all relevant (algorithmic) problems with finite structure semantics

Finite Model Theory

- Fundamental ideas
  1. Consider DBs as finite FOL structures
  2. Consider FOL as query language over DBs
- Starting with FOL investigate all relevant (algorithmic) problems with finite structure semantics
- These ideas make up an approximative but nonetheless very fruitful theoretical approach to studying DB related problems
  - Showing expressivity bounds for query languages
  - Showing equivalence of DB query languages
  - Showing the inherent complexity of DB query languages

FOL as a Query Language

- FOL query formula $\phi(\vec{x})$ (for $\vec{x} = x_1 \ldots, x_n$) over signature σ
  - $\vec{x}$ = distinguished variables, answer variables.

Definition (Answers of a query on a structure)

$$
\begin{aligned}
ans(\phi(\vec{x}), \mathfrak{A}) &= \mathfrak{A}^{\phi(\vec{x})} \\
&= \{ \vec{d} = (d_1, \ldots, d_n) \mid d_i \in A \text{ and } \mathfrak{A} \models \phi(\vec{x}/\vec{d}) \}
\end{aligned}
$$

- Set of answers can be considered as a structure with n-ary predicate ans
- n-ary query induced by ϕ:

$$Q_\phi : STRUCT(\sigma) \longrightarrow STRUCT(ans)$$

Boolean Queries

- Boolean FOL query formula = FOL formulae without free variables (also called sentences)
- According to definition possible answers are $\{()\}$ (stands for true) and $\emptyset$ (false)
- Boolean queries can be identified with the class of σ-structures making them true

Answering (Boolean) FOL queries

- Why is FOL so successful in DB theory?
- E.g., is model checking problem ($\mathfrak{A} \models \phi$) feasible?

Answering (Boolean) FOL queries

- Why is FOL so successful in DB theory?
- E.g., is model checking problem ($\mathfrak{A} \models \phi$) feasible?

- Answer is NOT really if considering $\mathfrak{A}, \phi$ both as inputs
  $\Longrightarrow$ Combined complexity

Theorem (Stockmeyer 74, Vardi 82)

Model-checking for FOL (and monadic second-order logic MSO) is PSPACE complete.

Lit: L. J. Stockmeyer. The complexity of decision problems in automata theory and logic. PhD thesis, MIT, 1974.

Lit: M. Y. Vardi. The complexity of relational query languages (extended abstract). In Proceedings of the Fourteenth Annual ACM Symposium on Theory of Computing, STOC '82, pages 137–146, New York, NY, USA, 1982. ACM.

Reminder: Complexity Classes

- Encode algorithmic problem Π as a language $\Pi \subseteq \Sigma^*$, i.e., as set of words over an alphabet Σ.
- Decision problem: $w \in \Pi$?

- Example complexity classes
  - PTIME = Problems solvable in polynomial time (w.r.t. the input size) by a deterministic Turing machine
  - PSPACE = Problems solvable in polynomial space (w.r.t. the input size) by a deterministic Turing machine

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- Usually, as computer scientist, you do not refer directly to TMs for getting complexity results
- Instead you (should) train yourself in the art of reducing and learning paradigmatic problems in complexity classes.

Lit: Complexity Zoo: `https://complexityzoo.uwaterloo.ca/Complexity_Zoo`

Lit: M. R. Garey and D. S. Johnson. Computers and Intractability; A Guide to the Theory of NP-Completeness. W. H. Freeman & Co., New York, NY, USA, 1990.

Complete Problems

- "Paradigmatic" problems in a complexity class $\mathcal{C}$ called $\mathcal{C}$-complete problems

- $\mathcal{C}$ complete problems (w.r.t. $\mathcal{C}'$ reductions)=
  Most difficult problems in $\mathcal{C}$ =
  $\{\Pi \mid \Pi \in \mathcal{C}$ and all other $\mathcal{C}$ problems are $\mathcal{C}'$-reducible to $\Pi\}$

- Problem $\Pi \subseteq \Sigma$ is $\mathcal{C}$-reducible to problem $\Pi' \subseteq \Sigma'$, for short: $\Pi \leq_C \Pi'$, iff there is a $\mathcal{C}$-computable function such that for all $w \in \Sigma^*$: $w \in \Pi$ iff $f(w) \in \Pi'$

Example for PSPACE Complete Problem

- Quantified Boolean Formula (QBF)
  - All propositional symbols p_i are QBF
  - All boolean combinations of QBFs are QBFs
  - If ϕ is a QBF, then so are $\forall p \phi$ and $\exists p \phi$.
  - Semantics: Structures here are truth value assignments

Theorem

Satisfiability of QBFs is PSPACE complete

Example

- $\exists p \exists q \ p \wedge q$ is satisfiable, because there is assignment $\nu(q) = 1$ and $\nu(p) = 1$ making $p \wedge q$ true.
- $\exists p \ p \wedge \neg p$ is not satisfiable

FOL is in PSPACE

Complexity estimation for query answering

Time complexity for checking $\mathfrak{A} \models \phi$ is $O(n^k)$, where

- n = size of input structure $\mathfrak{A}$ and
- k = size of input query ϕ

Reminder: Landau-Notation

- $f \in O(g)$ means: f has function g as upper bound
- Formally: There are constants $c > 0$ and x_0, s.t. for all $x > x_0$:

$$|f(x)| \leq c * |g(x)|$$

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Note: Size of query k is responsible for exponential blow up

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Complexity estimation for query answering

Time complexity for checking $\mathfrak{A} \models \phi$ is $O(n^k)$, where

- n = size of input structure $\mathfrak{A}$ and
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- Naive recursive algorithm showing time complexity of order $O(n^k)$ and space complexity of order $O(k * log(n))$
  - Atomic formula: Look up in structure
  - Boolean cases: apply semantics of Boolean connectors
  - $\exists x \phi(x)$: Check (until successful) for all $d \in A$ whether $\mathfrak{A} \models \phi(x/d)$

- PSPACE hardness by reducing QBF satisfiability to FOL model checking

FOL is in AC^0 in Data Complexity

- In practical scenarios DB size n much bigger than query size k
- Therefore: Consider only DB as input; query fixed
  $\Longrightarrow$ data complexity

- This helps a lot, as only query size responsible for exponential complexity, indeed:

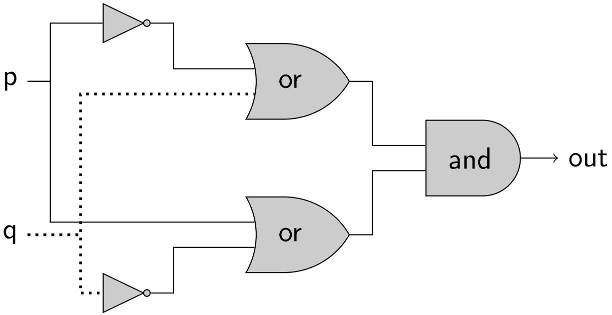
Theorem

Data complexity for FOL query answering is in LOGSPACE and even in AC^0.

- LOGSPACE = Problems solvable in logarithmic space on the read-write tape by a deterministic 2-tape Turing machine
- $AC^0 \subsetneq LOGSPACE$.

The Class AC^0

- Intuitively AC^0 = class of problems solvable in constant time on polynomially many processors (in parallel)
- Formally, AC^0 is defined using a computation model based on boolean circuits



Boolean circuit above computes $(\neg p \vee q) \wedge (p \vee \neg q)$

The class AC^0

- Encode problems as $0/1$ vector inputs
- Computability by circuits: There is (infinite) family of circuits (for every possible size of input) computing desired boolean function

- In many cases one also has a uniformity condition: family not arbitrarily constructed but computable as output of single TM

Definition

$AC^0 =$ Problems solvable by families of circuits with

- constant depth,
- polynomial size and
- using NOT gates, unlimited-fanin AND gates and OR gates.

FOL is in AC^0 data complexity

Proof idea

- Query modelled as boolean circuit family for every possible instance of given DB schema $\mathcal{R}$ and "super-domain" Dom
- Every ground atom $R(d_1, \ldots, d_n)$ is represented as propositional input symbol
- Gates for every subexpression of query
- Boolean operators in subexpression modelled by corresponding boolean gates
- $\exists$ ($\forall$) quantifier modelled by unbounded fan-in OR (AND) gate

Lit: S. Abiteboul, R. Hull, and V. Vianu. Foundations of Databases. 1995.

Proving Expressivity Bounds for FOL

Expressivity of Languages

- We defined above the query Q_ϕ induced by a formula ϕ (syntax $\rightarrow$ semantics direction)

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  - Given a query

$$Q : STRUC(\sigma) \longrightarrow STRUC(ans)$$

    test whether there is formula ϕ in the given logic s.t. $Q = Q_\phi$
  - In this case one says that Q is definable in the logic (for the given set of structures $STRUC(\sigma)$)

Expressivity of Languages

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    test whether there is formula ϕ in the given logic s.t. $Q = Q_\phi$
  - In this case one says that Q is definable in the logic (for the given set of structures $STRUC(\sigma)$)
- For Boolean queries definability amounts to:
  Given a class $X \subseteq STRUC(\sigma)$ of structures over a signature σ: there is a sentence ϕ (over the given logic) s.t. $Mod(\phi) = X$

Need for New Proof Techniques

Convention for the Following

All structures are finite, so

$$STRUC(\sigma) = \text{set of finite structures over } \sigma$$

- Main classical techniques used for classical FOL do not work
- Because corresponding theorems do not hold for FMT

- Reminder: Main properties of FOL
  - Compactness (Comp)
  - Löwenheim-Skolem (Löko)
- These properties characterize FOL for arbitrary structures: Lindström theorems

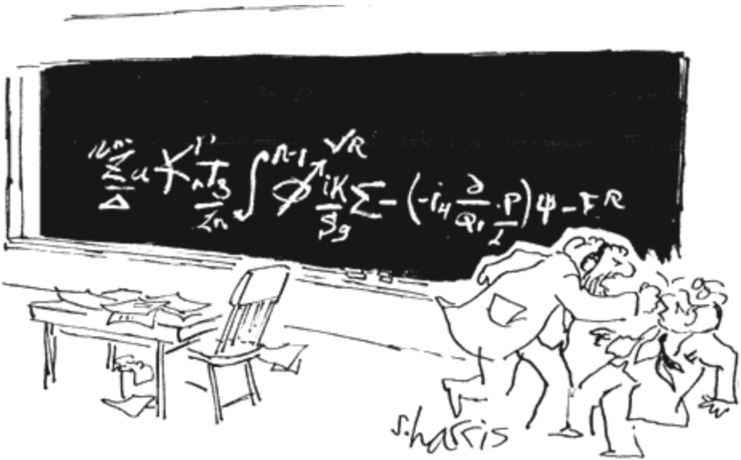
Finite Compactness Pendant?

Fin-Comp

If every finite subset of Φ has a finite model, then Φ has a finite model.

- The finite version of compactness (Fin-Comp) does not hold for FOL.
- Falsifier
  - $\lambda_n := \exists x_1, \ldots, x_n \bigwedge_{i \neq j} \neg(x_i = x_j)$
    (says: "There are at least n elements")
  - $\{\lambda_n \mid n \in \mathbb{N}\}$ has not finite model though every subset has

What's the Right "Proof Technique"?



"You want proof? I'll give you proof!"

We Prefer to Proof/Argue ... without Being a Poser ...

Pinguin Video

URL: `https://www.youtube.com/watch?v=7iDn5d9q9Y8`

Convention for the Following

Assume all structures are finite and relational, i.e., there are no function symbols other than constants—unless stated otherwise

Games

Games as Essence of Being a Human

"Der Mensch spielt nur, wo er in voller Bedeutung des Wortes Mensch ist, und er ist nur da ganz Mensch, wo er spielt"

(F. Schiller, Briefe Über die Ästhetische Erziehung des Menschen (1795))

Games as a CS Tool

- In logic, Fraïssé games are an important proof tool
- Different variations (w.r.t. rules, winning strategies)
- We will consider a basic game type and show how to use it.

- But: games have high "cognitive complexity" even for non-trivial problems
- Therefore: Use games for simple but generic problems and reduce others to these
  - Games have role similar to that of TMs

Ehrenfeucht-Fraïssé Games

- **Notation:** $G_n(\mathfrak{A}, \mathfrak{B})$
  n-round game played for structures on same signature
- **Input:** structures $\mathfrak{A}, \mathfrak{B}$
- **Players:** spoiler and duplicator
- **Output:** a function relating elements from $\mathfrak{A}$ with elements from $\mathfrak{B}$
- **Rules:** see next slide
- Spoiler's aim: show $\mathfrak{A}, \mathfrak{B}$ are "different"
- Duplicator's aim: show $\mathfrak{A}, \mathfrak{B}$ are "the same"

Rules of the Game

- In turn, spoiler choose structure and element i in it and
- duplicator chooses other structure and element in it
- After n rounds: n elements $a_1, \ldots, a_n$ from $\mathfrak{A}$ and n elements $b_1, \ldots, b_n$ from $\mathfrak{B}$ are chosen.

Winning condition

Duplicator wins iff
$(a_1, \ldots, a_n)$ plays in $\mathfrak{A}$ the same role as $(b_1, \ldots, b_n)$ in $\mathfrak{B}$

Partial Isomorphism

Formalize sameness of tuples' roles by notion of partial isomorphism

Definition (Partial Isomorphism)

For structures $\mathfrak{A}$, $\mathfrak{B}$ over signature σ, let $f : A \longrightarrow B$ be a (possibly partial) function with domain $dom(f)$. f is a partial isomorphism iff

- f is injective
- For every constant c: $c^{\mathfrak{A}} \in dom(f)$ and $f(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$
- For all (n-ary) relation symbols R (including identity) and all $a_1, \ldots, a_n \in dom(f)$
  $R^{\mathfrak{A}}(a_1, \ldots, a_n)$ iff $R^{\mathfrak{B}}(f(a_1), \ldots, f(a_n))$

If f is total and bijective, then f is called an isomorphism, and $\mathfrak{A}, \mathfrak{B}$ are said to be isomorphic, for short $\mathfrak{A} \simeq \mathfrak{B}$

Winning Condition Formalized

- After n rounds: up to n elements $a_1, \ldots, a_n$ from $\mathfrak{A}$ and up to n elements $b_1, \ldots, b_n$ from $\mathfrak{B}$ are chosen.
  (Note that we allow $a_i = a_j$)

- Winning condition
  Duplicator wins iff
  $f : a_i \mapsto b_i$ is a partial isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$.

- Game equivalence
  $\mathfrak{A} \sim_{G_n} \mathfrak{B}$ iff: Duplicator has a winning strategy in $G_n(\mathfrak{A}, \mathfrak{B})$
  ($\mathfrak{A}$ and $\mathfrak{B}$ are the same w.r.t. n-round games)

Quantifier Rank

- How do we use games for proving in-expressivity?

- We need two more technical notions
  1. to capture nesting depth of quantifiers
  2. to capture property that two structures model the same sentences (up to some syntactical complexity)

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Definition (Quantifier Rank $qr(\phi)$)

- $qr(\phi) = 0$ for atoms ϕ
- $qr(\phi \vee \psi) = qr(\phi \wedge \psi) = qr(\phi \to \psi) = max\{qr(\phi), qr(\psi)\}$
- $qr(\neg\phi) = qr(\phi)$
- $qr(\exists x\ \phi) = qr(\forall x\ \phi) = qr(\phi) + 1$

- Example: $qr(\ \forall x[\exists w(P(x, w)) \wedge \exists y \exists z R(x, y, z)]\ ) = 3$

Quantifier Rank

- How do we use games for proving in-expressivity?
- We need two more technical notions
  1. to capture nesting depth of quantifiers
  2. to capture property that two structures model the same sentences (up to some syntactical complexity)

Definition (Equivalence Up to Rank n)

$\mathfrak{A} \equiv_n \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ agree on all FOL sentences of quantifier rank up to n.

How to Use Games?

Theorem

$\mathfrak{A} \sim_{G_n} \mathfrak{B}$ *iff* $\mathfrak{A} \equiv_n \mathfrak{B}$

This gives a non-FOL-expressibility tool

- Aim: Show that boolean query Q is not expressible in FOL
- Construct families of structures $(\mathfrak{A}_n)_{n\in\mathbb{N}}, (\mathfrak{B}_n)_{n\in\mathbb{N}}$ s.t.
  1. All $\mathfrak{A}_n$ satisfy Q
  2. No $\mathfrak{B}_n$ satisfies Q
  3. $\mathfrak{A}_n \sim_{G_n} \mathfrak{B}_n$
- Assume Q expressible as FOL formula ϕ of quantifier rank n. Then $\mathfrak{A}_n \models \phi$ and $\mathfrak{B}_n \models \neg\phi$, but $\mathfrak{A}_n \sim_{G_n} \mathfrak{B}_n$ and by the theorem $\mathfrak{A}_n \equiv_n \mathfrak{B}_n$ ↯

Example: Inexpressibility of *EVEN*

- *EVEN*(σ): structures over signature σ with domain of even cardinality
- The signature is relevant for the proofs
- Simpel case: $\sigma = \emptyset \quad \Longrightarrow \quad$ structures are sets

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Proposition

EVEN$(\emptyset)$ *is not expressible in FOL*

Proof

- Choose $\mathfrak{A}_n$ as $2n$-element set, $\mathfrak{B}_n$ as $2n+1$-element set.
- $\mathfrak{A}_n \in \mathit{EVEN}(\emptyset)$ and $\mathfrak{B}_n \notin \mathit{EVEN}(\emptyset)$
- $\mathfrak{A}_n \sim_{G_n} \mathfrak{B}_n$: Duplicator plays already played element in the other set iff spoiler does

Inexpressibility of $EVEN(\sigma)$ with Games

- What about $EVEN(\sigma)$ for non-empty σ?
- Consider: $\sigma = \{<\}$ and class of structures = linear orders
- L_n: total ordering on set of n elements

Theorem

For every $m, k \geq 2^n$: $L_m \sim_{G_n} L_k$.

- (Not so simple) proof works with different sub-cases
- Corollary: $EVEN(<)$ not expressible over linear orders: take $\mathfrak{A}_n = L_{2^n}$, $\mathfrak{B}_n = L_{2^n+1}$.

Wake-Up Question

In the theorem before, why it is not sufficient to presume m, k which are not exponentially large in n in order to show that $L_m \sim_{G_n} L_k$?

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In the theorem before, why it is not sufficient to presume m, k which are not exponentially large in n in order to show that $L_m \sim_{G_n} L_k$?

Solution:

- The spoiler has n (= rounds) possibilities to dived the linear order (left or right). So there are at least 2^n possibilities.
- If $m \neq k$, say $m > k$, spoiler could go for the larger structure L_m and do division by halves on the larger part. At some point the duplicator will have less possibilities to choose a corresponding element.

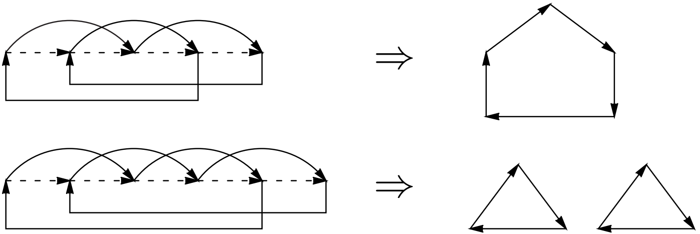
Proving Inexpressivity: Reduction Tricks (not Tools)

- Showing FOL inexpressibility of
  - graph connectivity *CONN*
  - acyclicity *ACYCL*
  - transitive closure *TC*

  by reduction of $\mathit{EVEN}(<)$ to each of them

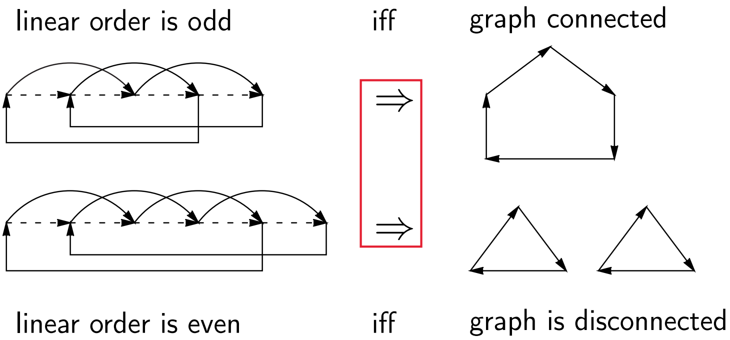
Reduce *EVEN*(<) to Graph Connectivity

linear order is odd iff graph connected



linear order is even iff graph is disconnected

Reduce $EVEN(<)$ to Graph Connectivity



- Construction of graph from linear order is expressible as an FOL query $Q_{red} : LinOrd \longrightarrow GRAPH$

A Very General Notion of Query

- In this discussion of the reduction of LinORD to CONN we use a very general notion of a FOL query. For completeness the exact definition is given below (See Immerman: Descriptive Complexity, p. 18)

Definition

Let τ, σ be two signatures with $\tau = (R_1^{a_1}, \ldots, R_r^{a_r}, c_1, \ldots, c_s)$ and k be a fixed natural number. A k-ary first-order query $Q : STRUCT(\sigma) \longrightarrow STRUCT(\tau)$ is given by an $r + s + 1$-tuple of σ-formulae $\phi_0, \phi_1, \ldots, \phi_r, \psi_1, \ldots, \psi_s$. For each σ-structure $\mathfrak{A} \in STRUC(\sigma)$ the formulae describe a τ-structure $Q(\mathfrak{A})$

$$Q(\mathfrak{A}) = (\ dom(Q(\mathfrak{A})), R_1^{Q(\mathfrak{A})}, \ldots, R_r^{Q(\mathfrak{A})}, c_1^{Q(\mathfrak{A})}, \ldots c_s^{Q(\mathfrak{A})}\ )$$

with

- $dom(Q(\mathfrak{A})) = \{(b^1, \ldots, b^k) \mid \mathfrak{A} \models \phi_0(b^1, \ldots, b^k)\}$
- $R_i^{Q(\mathfrak{A})} = \{(b_1^1, \ldots, b_1^k), \ldots, (b_i^1, \ldots, b_i^k) \in dom(Q(\mathfrak{A}))^{a_i} \mid \mathfrak{A} \models \phi_i(b_1^1, \ldots, b_{a_i}^k)\}$
- $c_j^{Q(\mathfrak{A})} =$ the unique $(b^1, \ldots, b^k) \in dom(Q(\mathfrak{A}))$ s.t. $\mathfrak{A} \models \psi_j(b^1, \ldots, b^k)$

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Example: $Q_{red} : LinOrd \rightarrow CONN$

- $\tau = E$, $\sigma =<$, $r = 1$, $s = 0$
- $k = 1$, $\phi_0 =$ an arbitrary tautology
- $\phi_1 =$ see the following slide

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- $dom(Q(\mathfrak{A})) = \{(b^1, \ldots, b^k) \mid \mathfrak{A} \models \phi_0(b^1, \ldots, b^k)\}$
- $R_i^{Q(\mathfrak{A})} = \{(b_1^1, \ldots, b_1^k), \ldots, (b_i^1, \ldots, b_i^k) \in dom(Q(\mathfrak{A}))^{a_i} \mid \mathfrak{A} \models \phi_i(b_1^1, \ldots, b_{a_i}^k)\}$
- $c_j^{Q(\mathfrak{A})} =$ the unique $(b^1, \ldots, b^k) \in dom(Q(\mathfrak{A}))$ s.t. $\mathfrak{A} \models \psi_j(b^1, \ldots, b^k)$

Example: $Q_{red} : LinOrd \rightarrow CONN$

- $\tau = E$, $\sigma = <$, $r = 1$, $s = 0$
- $k = 1$, $\phi_0 =$ an arbitrary tautology
- $\phi_1 =$ see the following slide

Note: k-arity does not talk about the number of answer variables—but the tuple size of elements over domain $\mathfrak{A}$

Finding the reduction

- Helper formulae
  - $succ(x, y) : x < y \land \neg\exists z.x < z \land z < y$
  - $last(x) : \neg\exists z.x < z$
  - $first(x) : \neg\exists z.z < x$
- Define Q_{red}: $LinOrd \longrightarrow GRAPH$ as

$$
\begin{aligned}
E(x, y) \quad = \quad & \psi(x, y) = \\
& (\exists z(succ(x, z) \land succ(z, y))) \lor \\
& (last(x) \land \exists z(first(z) \land succ(z, y))) \lor \\
& (\exists z(last(z) \land succ(x, z) \land first(y)))
\end{aligned}
$$

- Assume that $CONN$ is expressible as FOL query ϕ_{conn} over signature $\{E\}$ for graphs.
- Then $EVEN(<)$ would be FOL expressible as:
  $\phi_{conn}[E/\psi]$ ↯
  (Note: $\phi_{conn}[E/\psi]$ is shorthand for replacing every occurrence of atom $E(u, w)$ by formula $\psi(u, w)$ in ϕ_{conn}.)

ACYCL and TC are not FOL expressible

- *ACYCL*: Reduction $EVEN \Rightarrow ACYCL$ as above but with one back edge from last node to first node

- Reduction for *TC*: $CONN \Rightarrow TC$
  - Add edge $E(x, y)$ for every edge $E(y, x)$
  - Compute *TC* on resulting graph
  - Test whether graph is complete