UNIVERSITÄT ZU LÜBECK
INSTITUT FÜR INFORMATIONSSYSTEME

Özgür L. Özçep

Ontology-Based Data Access

Lecture 8: Motivation, Description Logics
4 June 2020

Informationssysteme CS4130
(Summer 2020)

References

- ESSLLI 2010 Course by Calvanese and Zakharyaschev

  http://www.inf.unibz.it/~calvanese/teaching/2010-08-ESSLLI-DL-QA/

- Reasoning Web Summer School 2014 course by Kontchakov on Description Logics

  http://rw2014.di.uoa.gr/sites/default/files/slides/An_Introduction_to_Description_Logics.pdf

- Lecture notes by Calvanese in 2013/2014 course on Ontology and Database Systems

  https://www.inf.unibz.it/~calvanese/teaching/14-15-odbs/lecture-notes/

- Course notes by Franz Baader on Description Logics
- Parts of Reasoning Web Summer School 2014 course by Ö. on Ontology-Based Data Access on Temporal and Streaming Data

  http://rw2014.di.uoa.gr/sites/default/files/slides/Ontology_Based_Data_Access_on_Temporal_and_Streaming_Data.pdf

Ontology-Based Data Access as Information Integration

- Data Integration can be considered as information integration purely on DB level
- OBDA can be considered as information integration using an ontology
- Bridges DB world (closes world assumption) and ontology world (open world assumption)

Closed World Assumption

- DB theory: closed-world assumption (CWA)
  - All and only those facts mentioned in DB hold.
- Simple form of uncertain knowledge expressed by NULLs
  - For one incomplete DB there are many completions
  - Nonetheless: Type information on attribute constrains the possible attribute instances
- In DE incompleteness generated by different schemata
  Flight scenario: Source DB had no flight number, whilst target DB has
  $\Longrightarrow$ introduction of NULLs for flight number attribute
- Logical theories (ontologies) adhere to open world assumption (OWA)
  - If something is not told, then we do not know
  - Logical theories (ontologies) may have many models

Close-World Assumption (CWA) for DBs

- "The world described by DBs is complete"

Example

University employee

ID	Name
1	Sokrates
2	Platon
3	Aristotle

Professor

ID
1
2

"3" (= ID of Aristotle) not in table Professor
$\Longrightarrow$ Aristotle is not a professor

Example

Patient

ID	Name
1	Sokrates
2	Platon
3	Aristotle

Blood sugar

ID	value
1	90
2	120

NULLs

- NULLs intended to model incompleteness
- but semantics not clear and hence highly criticized

Lit: L. Libkin. SQL's three-valued logic and certain answers. ACM Trans. Database Syst., 41(1):1:1–1:28, 2016.

Example

Patient

ID	Name
1	Sokrates
2	Platon
3	Aristotle

Blood sugar

ID	value [30-600]
1	90
2	120
3	NULL

Aristotle has a blood sugar value (30 or 31 or . . .)

Example

Patient

ID	Name
1	Sokrates

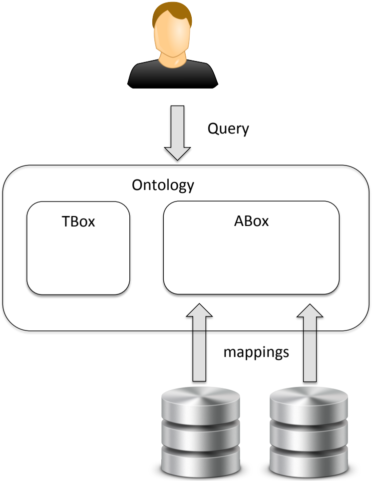
Pregnancy

ID	HCG value
1	NULL

OBDA: Motivation and Overview

Ontology-Based Data Access

- Use ontologies as interface
  …
- to access (here: query)
- data stored in some format
  …
- using mappings

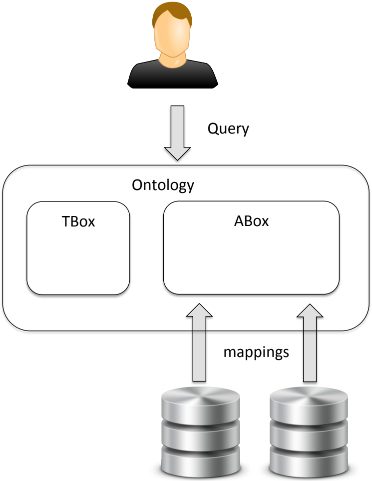


Ontology-Based Data Access

- Use ontologies as interface

  ...
- to access (here: query)
- data stored in some format

  ...
- using mappings



Ontologies

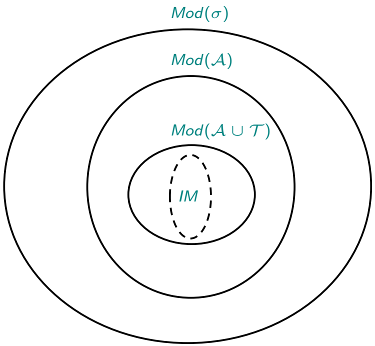
- Ontologies are triples of the form $\mathcal{O} = (\sigma, \mathcal{T}, \mathcal{A})$
  - Signature σ: Non-logical vocabulary $\sigma = N_i \cup N_C \cup N_R$
  - Tbox $\mathcal{T}$: set of σ-axioms in some logic to capture terminological knowledge
    This lecture: ontologies represented in Description Logics (DLs)
  - Abox $\mathcal{A}$: set of σ-axioms in (same logic) to capture assertional/contingential knowledge

- Note: Sometimes only tbox termed ontology

- Semantics defined on the basis of σ-interpretations $\mathcal{I}$
  - $\mathcal{I} \models Ax$ iff $\mathcal{I}$ makes all axioms in Ax true
  - $Mod(Ax) = \{\mathcal{I} \models Ax\}$

General Idea

- $\mathcal{A}$: Represents facts in domain of interest
- Open world assumption: $Mod(\mathcal{A})$ is not a singleton
- $\mathcal{T}$: Constrains $Mod(\mathcal{A})$ with intended σ readings
- Usually one has only approximations of intended models IM
- Realize inference services on the basis of the constrained interpretations



WARNING: A Misconception

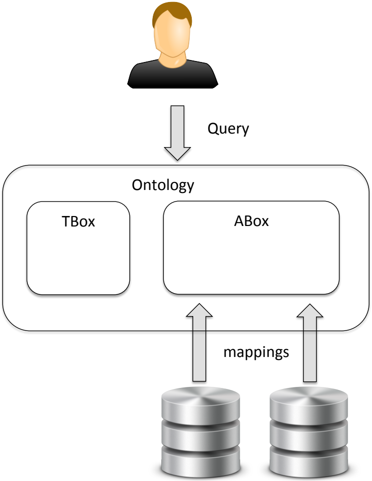
- With ontologies one does not declare data structures

- Abox data in most cases show pattern of data structures
- One does not have to re-model patterns/constraints in the abox data
  - Knowing "All A are B" in the abox is different from stipulating $A \sqsubseteq B$ (the former is known as integrity constraint)
  - Add $A \sqsubseteq B$, if you need to handle this relation for objects not mentioned in the abox

- Motto: Keep the tbox simple

Ontology-Based Data Access

- Use ontologies as interface
  ...
- to access (here: query)
- data stored in some format
  ...
- using mappings



Reasoning Services

- Different standard and nonstandard reasoning services exist
- May be reducible to each other

Example (Reasoning Services)

consistency check, subsumption check, taxonomy calculations, most specific subsumer, most specific concept, matching, ...

- In classical OBDA focus on
  - Consistency checking: $Mod(\mathcal{A} \cup \mathcal{T}) \neq \emptyset$.
  - Query answering
- Next to abox and tbox language query language QL over σ is a relevant factor for OBDA
- Certain query answering

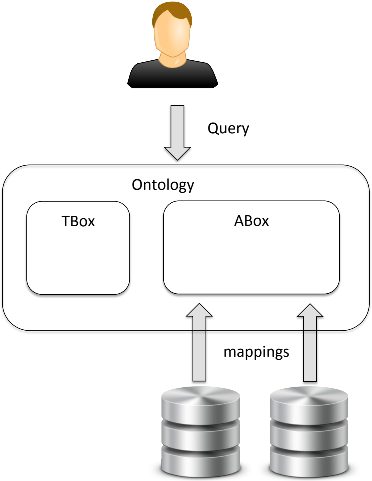
$$cert(\psi(\vec{x}), \mathcal{T} \cup \mathcal{A}) \quad = \quad \{\vec{a} \in (N_i)^n \mid \mathcal{T} \cup \mathcal{A} \models \psi[\vec{x}/\vec{a}]\}$$

Ontology-Based Data Access

- Use ontologies as interface

  ...
- to access (here: query)
- data stored in some format

  ...
- using mappings

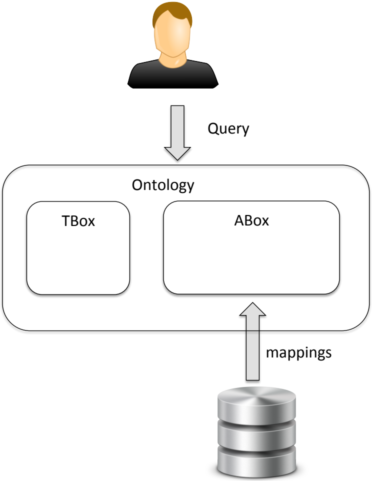


Backend Data Sources

- Classically: relational SQL DBs with static data
- Possible extensions: non-SQL DBs
  - datawarehouse repositories for statistical applications
  - pure logfiles
  - RDF repositories
- Non-static data
  - historical data (stored in temporal DB)
  - dynamic data coming in streams
- Originally intended for multiple DBs but ...

Ontology-Based Data Access

- Use ontologies as interface
  ...
- to access (here: query)
- data stored in some format
  ...
- using mappings



Mappings

- Mappings have an important crucial role in OBDA
- Lift data to the ontology level
  - Data level: (nearly) close world
  - Ontology Level: open world

Definition (Schema of Mappings)

$$m : \psi(\vec{f}(\vec{x})) \longleftarrow Q(\vec{x}, \vec{y})$$

- $\psi(\vec{f}(\vec{x}))$: Template (query) for generating abox axioms
- $Q(\vec{x}, \vec{y})$: Query over the backend sources
- Function $\vec{f}$ translates backend instantiations of $\vec{x}$ to constants

- Mappings M over backend sources generates abox $\mathcal{A}(M, DB)$.

Example Scenario: Measurements

- Example schema for measurement and event data in DB

```
SENSOR(SID, CID, Sname, TID, description)
SENSORTYPE(TID, Tname)
COMPONENT(CID, superCID, AID, Cname)
ASSEMBLY(AID, AName, ALocation)
MEASUREMENT(MID, MtimeStamp, SID, Mval)
MESSAGE(MesID, MesTimeStamp, MesAssemblyID,  catID, MesEventText)
CATEGORY(catID, catName)
```

- For mapping

m: $\mathit{Sens}(x) \wedge \mathit{name}(x, y) \longleftarrow$

```
SELECT f(SID) as x,  Sname as y FROM SENSOR
```

- the row data in SENSOR table

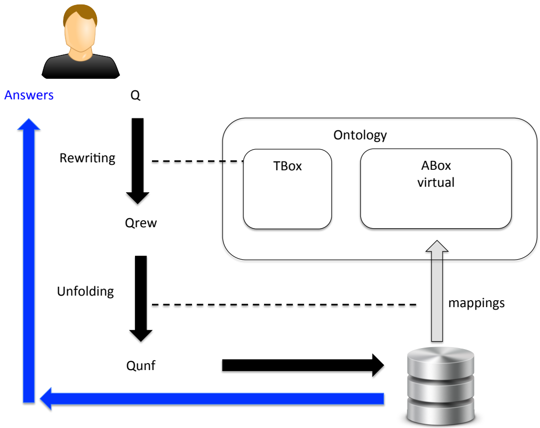
SENSOR
(123, comp45, TempSens, TC255, 'A temperature sensor')

- generates facts

$\mathit{Sens}(f(123)), \mathit{name}(f(123), \mathit{TempSens}) \in \mathcal{A}(m, DB)$

OBDA in the Classical Sense

- Keep the data where they are because of large volume
- Abox is virtual (no materialization)
- First-order logic (FOL) perfect rewriting + unfolding



Ontologies and Description Logics

Description Logics

Definition

Description logics (DLs) are logics for use in knowledge representation with special attention on a good balance of expressibility and feasibility of reasoning services

- Can be mapped to fragments of FOL
- Use
  - as ontology representation language for conceptual modeling
  - in particular in the semantic web
  - Formal counterpart of standard web ontology language (OWL)
  - and in particular for ontology-based data access (OBDA)
- Have been investigated for ca. 30 years now
  - Many theoretical insights on various different purpose DLs
  - General-purpose reasoners (RacerPro, Fact++, ...) and specific reasoners (Quest,...)
  - Various editing tools (most notably Protege)

Family of DLs

- Variable-free logics centered around concepts
- concepts = one-ary predicates in FOL = classes in OWL

Example (Concepts)

- $Students$ ("students")
- $Students \sqcap Male$ (" Male students")
- $\exists attends.MathCourse$ ("Those attending a math course")
- $\forall hasFriends.Freaks$ ("Those having only freaks as friends")
- $Person \sqcap \forall attends.(Course \sqcap \neg easy)$ ("Persons attending only non-easy courses")

An (Semi-)Expressive Logic: $\mathcal{ALC}$

- Vocabulary: constants N_i, atomic concepts N_C, roles N_R

- Concept(description)s: syntax

$$C \quad ::= \quad A \quad (\text{for } A \in N_C) \mid C \sqcap C \mid C \sqcup C \mid \neg C \mid \forall r.C \mid \exists r.C \; (\text{for } r \in N_R) \mid \bot \mid \top$$

- Concept(description)s: semantics

- Interpretation $\mathcal{I} = (\underbrace{\Delta^{\mathcal{I}}}_{\text{domain}}, \overbrace{\cdot^{\mathcal{I}}}^{\text{denotation function}})$
- $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ for all $A \in N_C$
- $c^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ for all $c \in N_i$
- $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for all $r \in N_r$

- $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$
- $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$
- $\neg C = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
- $(\forall r.C)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid$ for all $e \in \Delta^{\mathcal{I}}$: If $(d, e) \in r^{\mathcal{I}}$ then $e \in C^{\mathcal{I}}\}$
- $(\exists r.C)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid$ there is $e \in \Delta^{\mathcal{I}}$ s.t. $(d, e) \in r^{\mathcal{I}}$ and $e \in C^{\mathcal{I}}\}$

Tbox and Abox

- Terminological Box (tbox) $\mathcal{T}$
  - Finite set of general concept inclusions (GCIs)
  - GCI: axioms of form $C \sqsubseteq D$ (for arbitrary concept descriptions) $C \equiv D$ abbreviates $\{C \sqsubseteq D, D \sqsubseteq C\}$
  - Semantics: $\mathcal{I} \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

- Assertional Box (abox) $\mathcal{A}$
  - Finite set of assertions
  - Assertion: $C(a)$ (concept assertion), $r(a, b)$ (role assertion)
  - Semantics:
    $\mathcal{I} \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$
    $\mathcal{I} \models r(a, b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$.

- Ontology: $(\sigma, \mathcal{T}, \mathcal{A})$ or just $\mathcal{T} \cup \mathcal{A}$

Example (University)

$$
\begin{aligned}
\mathcal{T} &= \{\ GradStudent \sqsubseteq Student, \\
&\quad GradStudent \sqsubseteq \exists takesCourse.GradCourse\ \} \\
\mathcal{A} &= \{\ GradStudent(john)\ \}
\end{aligned}
$$

Consider the following interpretations

- $\mathcal{I}_1$:
  - $john^{\mathcal{I}_1} = j$
  - $GradStudent^{\mathcal{I}_1} = \{j\}$
  - $Student^{\mathcal{I}_1} = \{j\}$
  - $GradCourse^{\mathcal{I}_1} = \{s\}$
  - $takesCourse^{\mathcal{I}_1} = \{(j, s)\}$
- $\mathcal{I}_1 \models \mathcal{T} \cup \mathcal{A}$
- $\mathcal{I}_2$:
  - $john^{\mathcal{I}_2} = j$
  - $GradStudent^{\mathcal{I}_2} = \{j\}$
  - $Student^{\mathcal{I}_2} = \{j\}$
  - $GradCourse^{\mathcal{I}_2} = \{j\}$
  - $takesCourse^{\mathcal{I}_2} = \{(j, j)\}$
- $\mathcal{I}_2 \models \mathcal{T} \cup \mathcal{A}$
- $\mathcal{I}_3$:
  - $john^{\mathcal{I}_1} = j$

Stricter notion of Tbox

- Above definition of tbox very general
  - "Meanings" of concept names determined only implicitly in the whole ontology
  - No guarantee for unique extensions

- Early notion of tbox more related to idea of explicitly defining concept names
- $C \equiv D$ used as abbreviation for $C \sqsubseteq D$ and $D \sqsubseteq C$
- Concept definition: $A \equiv D$ (where A atomic)

Definition

A TBox in a strict sense is a finite set of concept definitions not defining a concept multiple times or in a cyclic manner. Defined concepts occur on the lhs, primitive concept on the rhs of definitions.

Digression: Implicit vs. Explicit Definability

- Sometimes a general tbox may fix the denotation of a concept name w.r.t. denotations of the others $\Longrightarrow$ implicit definability
- Maybe then it can also be defined explicitly?

Definition

Given an FOL theory Ψ over signature σ and a predicate symbol R.

- R is implicitly defined in Ψ iff for any two models $\mathfrak{A} \models \Psi$ and $\mathfrak{B} \models \Psi$ agreeing on $\sigma \setminus \{R\}$ one has $R^{\mathfrak{A}} = R^{\mathfrak{B}}$.
- R is explicitly defined in Ψ by a formula $\phi(\vec{x})$ not containing R iff $\Psi \models \forall \vec{x} R(\vec{x}) \leftrightarrow \phi(\vec{x})$

Digression: Beth Definability Theorem

- For FOL both implicit and explicit definability coincide

Theorem

An FOL theory defines a predicate implicitly iff it defines it explicitly

- Though DLs can be embedded into FOL, the equivalence of implicit and explicit definability does not transfer necessarily to DLs
- At least it does for $\mathcal{ALC}$ theories

  Lit: B. ten Cate, E. Franconi, and I. Seylan. Beth definability in expressive description logics. J. Artif. Int. Res., 48(1): 347–414, Oct. 2013.

Reasoning services

- Semantical notions as in FOL but additional notions due to focus on concepts
- Let $\mathcal{O} = (\sigma, \mathcal{T}, \mathcal{A})$

Definition (Basic Semantical Notions)

- Model: $\mathcal{I} \models \mathcal{O}$ iff $\mathcal{I} \models \mathcal{T} \cup \mathcal{A}$
- Satisfiability: $\mathcal{O}$ is satisfiable iff $\mathcal{T} \cup \mathcal{A}$ is satisfiable
- Coherence: $\mathcal{O}$ is coherent iff $\mathcal{T} \cup \mathcal{A}$ has a model $\mathcal{I}$ s.t. for all concept names $A^{\mathcal{I}} \neq \emptyset$
- Concept satisfiability: C is satisfiable w.r.t. $\mathcal{O}$ iff there is $\mathcal{I} \models \mathcal{O}$ s.t. $C^{\mathcal{I}} \neq \emptyset$
- Subsumption: C is subsumed by D w.r.t. $\mathcal{O}$ iff $\mathcal{O} \vDash C \sqsubseteq D$ iff $\mathcal{T} \cup \mathcal{A} \vDash C \sqsubseteq D$
- Instance check: a is an instance of C w.r.t. $\mathcal{O}$ iff $\mathcal{O} \vDash C(a)$

Reduction Examples

- Many of the semantical notions are reducible to each other
- We give only one example

Exercise

Show that subsumption can be reduced to satisfiability tests (allowing the introduction of new constants). More concretely:

$C \sqsubseteq D$ w.r.t. $\mathcal{O}$ iff $(\sigma \cup \{b\}, \mathcal{T}, \mathcal{A} \cup \{C(b), \neg D(b)\})$ is not satisfiable (where b is a fresh constant).

Extended Reasoning Services

Definition

- Instance retrieval: Find all constants x s.t. $\mathcal{O} \models C(x)$
- Query answering: Certain answers
  $cert(\phi(x), \mathcal{O}) = \{\vec{a} \in N_i \mid \mathcal{O} \models \phi[\vec{x}/\vec{a}]\}$
- Classification: Compute the subsumption hierarchy of all concept names
- Realization: Compute the most specific concept name to which a given constant belongs
- Pinpointing, matching, ...

Example (Certain Answers for Conjunctive Queries)

$$\mathcal{T} = \{ \top \sqsubseteq \mathit{Male} \sqcup \mathit{Female}, \mathit{Male} \sqcap \mathit{Female} \sqsubseteq \bot \}$$

$$\mathcal{A} = \{ \mathit{friend}(\mathit{john}, \mathit{susan}), \mathit{friend}(\mathit{john}, \mathit{andrea}), \mathit{female}(\mathit{susan}), \mathit{likes}(\mathit{susan}, \mathit{andrea}), \mathit{likes}(\mathit{andrea}, \mathit{bill}), \mathit{Male}(\mathit{bill}) \}$$

$$Q(x) = \exists y, z(\mathit{friend}(x, y) \wedge \mathit{Female}(y) \wedge \mathit{likes}(y, z) \wedge \mathit{Male}(z))$$

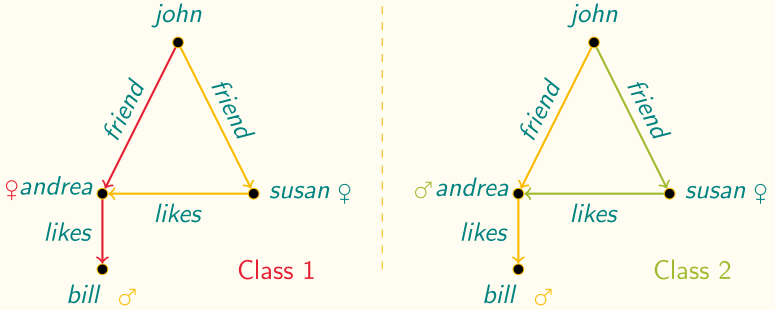
- $\mathit{cert}(Q(x), \mathcal{O}) =?$

- We have to consider all possible models of the ontology
- But here there are actually two classes:
  Andrea is male vs. Andrea is not male.

Example (Certain Answers for Conjunctive Queries)

$$\mathcal{T} = \{ \top \sqsubseteq Male \sqcup Female, Male \sqcap Female \sqsubseteq \bot \}$$
$$\mathcal{A} = \{ friend(john, susan), friend(john, andrea), female(susan), likes(susan, andrea), likes(andrea, bill), Male(bill) \}$$

$$Q(x) = \exists y, z(friend(x, y) \land Female(y) \land likes(y, z) \land Male(z))$$



$$cert(Q(x), \mathcal{O}) = \{john\}$$

Embedding into FOL

- Most DLs (such as $\mathcal{ALC}$) can be embedded into FOL
- Notion of embedding is well-defined as FOL structures are used for semantics of DLs.

- Correspondence idea
  Concept names = unary predicates, roles = binary predicates, GCI = $\forall$ rules
- Define for any concept description and variable x its corresponding x-open formula $\tau_x(C)$
  - $\tau_x(A) = A(x)$
  - $\tau_x(C \sqcap D) = \tau_x(C) \wedge \tau_x(D)$
  - $\tau_x(C \sqcup D) = \tau_x(C) \vee \tau_x(D)$
  - $\tau_x(\neg C) = \neg \tau_x(C)$
  - $\tau_x(\forall r.C) = \forall y(r(x,y) \rightarrow \tau_y(C))$
  - $\tau_x(\exists r.C) = \exists y(r(x,y) \wedge \tau_y(C))$
- Abox axioms not changed
- Tbox axioms: $C \sqsubseteq D$ becomes $\forall x(\tau_x(C) \rightarrow \tau_x(D))$

Embedding into FOL

- For translation two variables are sufficient ("2 finger movement")
- Hence: DLs embeddable into known 2-variable fragment of FOL
- Also the fragment is a guarded fragment: one quantifies over variables fixed within atom.

DL Family

- Different DLs for different purposes
  - What is more important: Expressivity or feasibility?
  - Which kinds of reasoning services does one have to provide?

- Differences regarding
  - the allowed set of concept constructors
  - the allowed set of role constructors
  - the allowed types of tbox axioms
  - the allowed types of abox axioms
  - the allowance of concrete domains and attributes (such as *hasAge* with range the domain of integers)

Family of DLs and their Namings

- $\mathcal{AL}$: attributive language
- $\mathcal{C}$: (full) complement/negation
- $\mathcal{I}$: inverse roles $((r^{-1})^{\mathcal{I}} = \{(d,e) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (e,d) \in r^{\mathcal{I}}\})$
- $\mathcal{H}$: role inclusions $(\mathit{hasFather} \sqsubseteq \mathit{hasParent})$

- $\mathcal{S}$: $\mathcal{ALC}$ + transitive roles $(\mathit{trans\ isReachable})$
- $\mathcal{N}$: unqualified number restrictions
  $((\geq n\ r)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid \#(\{e \mid (d,e) \in r^{\mathcal{I}}\}) \geq n\})$
- $\mathcal{O}$: nominals $\{b\}^{\mathcal{I}} = \{b^{\mathcal{I}}\}$
- $\mathcal{Q}$: qualified number restrictions
  $((\geq n\ r.C)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid \#(\{e \mid (d,e) \in r^{\mathcal{I}}\}$ and $e \in C^{\mathcal{I}}) \geq n\})$
- $\mathcal{F}$: functionality constraints $\mathcal{I} \models (\mathit{func}\ R)$ iff $R^{\mathcal{I}}$ is a function
- $\mathcal{R}$: role chains and $\exists R.\mathit{Self}$ $(\mathit{hasFather} \circ \mathit{hasMother} \sqsubseteq \mathit{hasgrandMa})$
  $(\mathit{narcist} \equiv \exists \mathit{likes}.\mathit{Self})$

- OWL 2 is $\mathcal{SROIQ}$

Lightweight DLs

- Lightweight DLs favor feasibility over expressibility by, roughly, disallowing disjunction
- In principle three lightweight logics that have corresponding OWL 2 profiles (https://www.w3.org/TR/owl2-profiles/)

- $\mathcal{EL}$ (OWL 2 EL)
  - No inverses, no negation, no $\forall$
  - polynomial time algorithms for all the standard reasoning tasks with large ontologies
- DL-Lite (OWL 2 QL)
  - Tbox: No qualified existentials on lhs
  - Feasible CQ answering using rewriting and unfolding leveraging RDBS technology
- RL (OWL 2 RL)
  - Tbox restriction: "Only concept names on the rhs"
  - Polynomial time algorithms leveraging rule-extended database technologies operating directly on RDF triples

Comparison

	RL	EL	QL
inverse roles	+	-	+
rhs qual. exist	-	+	+
lhs qual. exist.	+	+	-

Complexity

- A nearly complete picture of reasoning services for DLs
- Research in DL community as of now resembles complexity farming
- DL complexity navigator:
  http://www.cs.man.ac.uk/~ezolin/dl (Last update 2013)

Tableaux Calculus for $\mathcal{ALC}$

- Efficient calculi are at the core of DL reasoners
- Tableaux calculi have been implemented successfully
- Refutation calculus based on disjunctive normal form
- We demonstrate it here at an example for $\mathcal{ALC}$ tboxes
- For a full description and proofs see handbook article by Baader

  Lit: F. Baader and W. Nutt. Basic description logics. In F. Baader et al., editors, The Description Logic Handbook, pages 43–95. Cambridge University Press, 2003.

Tableaux Example

- $\mathcal{ALC}$ tableau gives tests for satisfiability of abox
- by checking whether obvious contradictions (clashes with complementary literals) are contained
- An abox that is complete (no rules applicable anymore) and open (no clashes) describes a model
- Algorithm applies tableau rules to extend abox

Rules

- Starts with an abox $\mathcal{A}_0$ which is in negation normal form (NNF): all $\neg$ appear only in front of concept names)
- Apply rules to construct new aboxes; indeterminism due to $\sqcup$ rule

Rule	Condition	$\rightsquigarrow$	Effect
$\rightsquigarrow_\sqcap$	$(C \sqcap D)(x) \in \mathcal{A}$	$\rightsquigarrow$	$\mathcal{A} \cup \{C(x), D(x)\}$
$\rightsquigarrow_\sqcup$	$(C \sqcup D)(x) \in \mathcal{A}$	$\rightsquigarrow$	$\mathcal{A} \cup \{C(x)\}$ or $\mathcal{A} \cup \{D(x)\}$
$\rightsquigarrow_\exists$	$(\exists r.C)(x) \in \mathcal{A}$	$\rightsquigarrow$	$\mathcal{A} \cup \{r(x,y), C(y)\}$ for fresh y
$\rightsquigarrow_\forall$	$(\forall r.C)(x), r(x,y) \in \mathcal{A}$	$\rightsquigarrow$	$\mathcal{A} \cup \{C(y)\}$

- Rules applicable only if they lead to an addition of assertions
- One obtains a tree with aboxes (due to indeterminism)
- Within each abox a tree-like structure is established (tree-model property)

Example

- Given: $\mathcal{T} = \{GoodStudent \equiv Smart \sqcap Studious\}$
- Subsumption test:
  $\mathcal{T} \models \exists knows.Smart \sqcap \exists knows.Studious \sqsubseteq \exists knows.GoodStudent$
- Reduction to abox satisfiability:
  $\{\exists knows.Smart \sqcap \exists knows.Studious \sqcap \neg(\exists knows.GoodStudent)(a)\}$ satisfiable?
- Expansions of definition
  $\{\exists knows.Smart \sqcap \exists knows.Studious \sqcap \neg(\exists knows.(Smart \sqcap Studious))(a)\}$ satisfiable?
- Transform to NNF
  $\{\exists knows.Smart \sqcap \exists knows.Studious \sqcap \forall knows.(\neg Smart \sqcup \neg Studious)(a)\}$ satisfiable?

- Expansion step assumes a tbox in a strict sense, but GCIs can be transformed to definitions (i.e., axioms of the form $A \equiv C$) using additional symbols

Example (A Tableau Derivation)

- $\{\exists knows.Smart \sqcap \exists knows.Studious \sqcap \forall knows.(\neg Smart \sqcup \neg Studious)(a)\}$
- Abbreviation: $\{\exists r.A \sqcap \exists r.B \sqcap \forall r.(\neg A \sqcup \neg B)(a)\}$

$$\mathcal{A}_0 = \exists r.A \sqcap \exists r.B \sqcap \forall r.(\neg A \sqcup \neg B)(a)$$

$\rightsquigarrow_{\sqcap}$ (2 times)

$$\mathcal{A}_1 = \mathcal{A}_0 \cup \{(\exists r.A)(a),\ (\exists r.B)(a),\ (\forall r.(\neg A \sqcup \neg B))(a)\}$$

$\rightsquigarrow_{\exists}$ (2 times)

$$\mathcal{A}_2 = \mathcal{A}_1 \cup \{r(a,b), A(b), r(a,c), B(c)\}$$

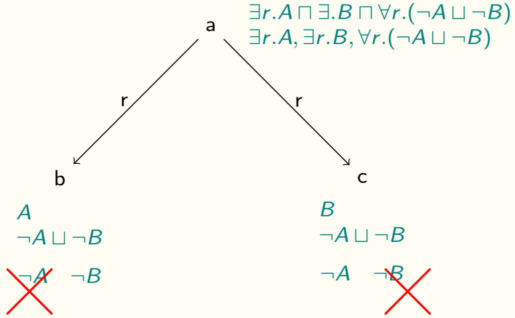
$\rightsquigarrow_{\forall}$ (2 times)

$$\mathcal{A}_3 = \mathcal{A}_2 \cup \{(\neg A \sqcup \neg B)(b), (\neg A \sqcup \neg B)(c)\}$$

- $\rightsquigarrow_{\sqcup}$ $\mathcal{A}_{4.1} = \mathcal{A}_3 \cup \{\neg A(b)\}$
  - $\rightsquigarrow_{\sqcup}$ $\mathcal{A}_{5.11} = \mathcal{A}_{4.1} \cup \{\neg A(c)\}$ — clash
  - $\rightsquigarrow_{\sqcup}$ $\mathcal{A}_{5.12} = \mathcal{A}_{4.1} \cup \{\neg B(c)\}$ — clash
- $\rightsquigarrow_{\sqcup}$ $\mathcal{A}_{4.2} = \mathcal{A}_3 \cup \{\neg B(b)\}$
  - $\rightsquigarrow_{\sqcup}$ $\mathcal{A}_{5.21} = \mathcal{A}_{4.2} \cup \{\neg A(c)\}$
  - $\rightsquigarrow_{\sqcup}$ $\mathcal{A}_{5.22} = \mathcal{A}_{4.2} \cup \{\neg B(c)\}$ — clash

Example (The partial tree model in the aboxes)

- $\{\exists knows.Smart \sqcap \exists knows.Studious \sqcap \forall knows.(\neg Smart \sqcup \neg Studious)(a)\}$
- Abbreviation: $\{\exists r.A \sqcap \exists r.B \sqcap \forall r.(\neg A \sqcup \neg B)(a)\}$



- Canonical tree model(s) can be directly read off (here from abox $\mathcal{A}_{5.21}$ which contains no clash):
  $\mathcal{I} = (\{a, b, c\}, \cdot^{\mathcal{I}})$ with
  $r^{\mathcal{I}} = \{(a, b), (a, c)\}$ $\quad A^{\mathcal{I}} = \{b\}$ $\quad B^{\mathcal{I}} = \{c\}$

Tableaux Calculus

- The tableau calculus for $\mathcal{ALC}$ is complete, correct, and terminates.
- Hence, the following properties hold

Theorem

- *$\mathcal{ALC}$ abox satisfiability (concept satisfiability, subsumption...) is decidable*
- *$\mathcal{ALC}$ has the finite model property, i.e.
if an $\mathcal{ALC}$ ontology has a model, then it has a finite model.*
- *$\mathcal{ALC}$ has the tree model property*