Intelligent Agents

Doxastic logic and dynamics of beliefs

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Todays lecture based on

- Parts of Lecture notes „EPISTEMIC LOGICS“ by Andreas Herzig, 2017
  https://www.irit.fr/~Andreas.Herzig/Cours/epiLogics.pdf

DOXASTIC LOGIC

Relevance of Knowledge

- When is knowledge the appropriate informational attitude?
- Remember: “knowledge entails truth” principle in epistemic logic: $\vDash_{S5n}\ K_a\phi \rightarrow \phi$
- Relevant for:
  - formal epistemology
    - What is knowledge?
    - Is knowledge possible at all?
    - Are all truths knowable?
  - Distributed processes (Fagin et al 03)

Truth

- Relation to truth less in focus in:
  - philosophy of mind: focus on agent's mental state
  - philosophy of language: effects of speech acts on the participants' mental states: lies, bullshitting
  - implementation of artificial agents
- informational mental attitude not entailing truth: belief
  - "he knows that ϕ, but he is wrong": inconsistent
  - "he believes that ϕ, but he is wrong": consistent
  - however: 'belief aims at truth' (Engel 1998), (Hakli 2006)
- Doxastic logic (Hintikka 2005) (Lenzen 1978, Lenzen 1995)
  - doxa = δοξα = 'belief' (Greek)

Definition (Syntax of Doxastic Logic: KD45n)

- Well-formed formula of doxastic logic are given by BNF:

$$\phi ::= p \mid \bot \mid \neg\phi \mid (\phi \wedge \phi) \mid B_a\phi$$

  where $p \in AP$ and $a \in AGT$.
- Intended reading: $B_a\phi$ ``agent a believes ϕ“
- Dual operator: $\widehat{B}_a$ abbreviates $\neg B_a \neg\phi$

  ``it is possible for a that ϕ“

Example

- $p \wedge B_a \neg p$
- $\mathrm{B}_\mathrm{a} \neg p \wedge B_b B_a p$
- $B_a(B_b p \vee B_b \neg p)$

Doxastic attitudes and situations

- Three possible doxastic attitudes w.r.t. a formula ϕ

$$B_a\phi \quad \hat{B}_a\phi \wedge \hat{B}_a\neg\phi \quad B_a\neg\phi$$

  for ϕ contingent (not tautology and not contradiction) and non-doxastic
- Six possible doxastic situations w.r.t. a formula ϕ

$$\phi \wedge B_a\phi \qquad \phi \wedge \hat{B}_a\phi \wedge \hat{B}_a\neg\phi \qquad \phi \wedge B_a\neg\phi$$

$$\neg\phi \wedge B_a\phi \quad \neg\phi \wedge \hat{B}_a\phi \wedge \hat{B}_a\neg\phi \quad \neg\phi \wedge B_a\neg\phi$$

  for ϕ contingent (not tautology and not contradiction) and non-doxastic

Semantics

Belief explained (as for knowledge) with possible worlds

$B_a\phi$ = „agent a believes that ϕ"

=„ϕ true in every world that is compatible with a's beliefs"

Definition (Models of KD45n)

A $KD45_n$-model is a structure $\mathcal{M} = (W, B, V)$ where

- W nonempty set (of possible worlds)
- $V: AP \rightarrow 2^W$ (valuation)
- $\mathcal{R}: AGT \rightarrow 2^{W \times W}$ such that for every $a \in AGT$:
  - For every w there is some w' such that $(w, w') \in \mathcal{R}_a$ (serial)
  - If $(w, w') \in \mathcal{R}_a$ and $(w', w'') \in \mathcal{R}_a$, then $(w, w'') \in \mathcal{R}_a$ (transitive)
  - If $(w, w') \in \mathcal{R}_a$ and $(w, w'') \in \mathcal{R}_a$, then $(w', w'') \in \mathcal{R}_a$ (Euclidean)

Some derived notions and observations

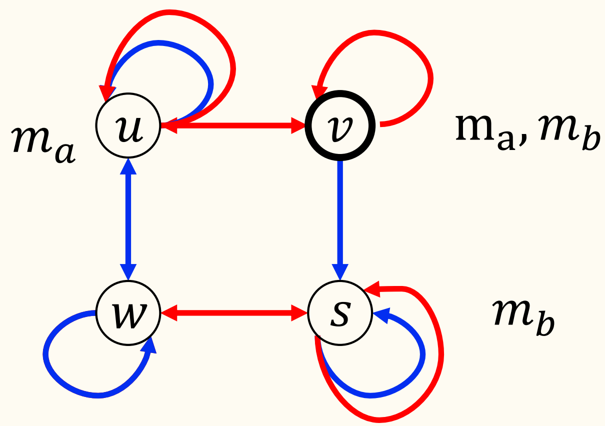
- $\mathcal{R}_a(w) = \{ w' \mid (w, w') \in \mathcal{R}_a \}$
  - $=$ a's alternatives to w
  - $=$ worlds a cannot distinguish from w on basis of beliefs
  - $=$ set of worlds compatible with a's beliefs
  - $=$ belief state of agent a at w
- $\mathcal{R}_a$ serial iff $\mathcal{R}_a(w) \neq \emptyset$
- $\mathcal{R}_a$ transitive and Euclidean iff:
  if $w' \in \mathcal{R}_a(w)$ then $\mathcal{R}_a(w) = \mathcal{R}_a(w')$

Definition (modellig relation in KD45n)

$\mathcal{M}, w \vDash B_a\phi$ iff $\mathcal{M}, w' \vDash \phi$ for every $w' \in \mathcal{R}_a(w)$

Example (Variant of muddy children (a, b) with beliefs)

Child a wrongly believes it is not muddy



$R_a(v) = \{s\}$

$M, v \vDash m_a \wedge B_a \neg m_a$

Axiomatics

Definition (A calculus for multimodal KD45n)

- Axioms for multimodal K
  - Axioms for propositional logic
  - Axiom $\mathrm{B}_a\phi \wedge B_a\psi \rightarrow B_a(\phi \wedge \psi)$ Axiom M(B_a)
  - Rule: $\phi \rightarrow \psi \quad \vdash_{KD45_n} \quad B_a\phi \rightarrow B_a\psi$
- Consistency of Belief: $\neg(B_a\, \phi \wedge B_a\neg\phi)$ Axiom $\mathrm{D}(B_a)$
- Positive Introspection: $B_a\phi \rightarrow B_aB_a\phi$ Axiom $4(B_a)$
- Negative Introspection: $\neg B_a\phi \rightarrow B_a\neg B_a\phi$ Axiom $5(B_a)$

Axiomatics

Theorem (Properties of calculus)

- Sound and complete: $\vdash_{KD45_n}$ iff $\vDash_{KD45_n}$
- Decidable
- Complexity of $KD45_n$-satisfiability
  - NP-complete if $n = 1$
  - PSPACE-complete if $n > 1$
- For $n = 1$ there exists a normal form: modal depth ≤ 1

Discussion: Omniscience problem

- Closure of B_a under inference (see only rule in calculus)
- This is not realistic - in particular for ressource bounded agents.

- (Negative) Introspection also criticised (Lenzen 78)

Discussion: belief and probability

- $KD45_n$'s notion of belief is strong („conviction")
- Weaker version:
  - $\mathrm{B}_a\phi = Prob_a(\phi) > Prob_a(\neg\,\phi)$
  - For classical semantics this amounts to $Prob_a(\phi) > \frac{1}{2}$
- Semantics: $\mathcal{M} = (W, \mathcal{R}, V)$ where

  $$\mathcal{R}: AGT \rightarrow (W \times W)$$

- $\mathcal{M}, w \vDash B_a\phi$ iff among the a-accessible worlds there are more ϕ than $\neg\phi$ worlds
  - $(B_a\phi \wedge B_a\psi) \rightarrow B_a(\phi \wedge \psi)$ not valid!
  - Weakening of Kripke semantics: neighbourhood semantics (Burgess 1969), (Lenzen 1978)

Discussion: Graded Belief

- Language: $B_a^{\geq d}\phi =$ „a believes ϕ with degree at least d
  (where $d \in [0,1]$)
- Semantics: $\mathcal{M} = (W, \mathcal{R}, V)$ where
  - $\mathcal{R}$: $AGT \times [0,1] \rightarrow (W \times W)$ such that $B_a^{\geq d}\phi \subseteq B_a^{\geq d+d'}\phi$
    Linear chain of accesibility relations
    (-> „system of spheres“)
  - $wB_a^{\geq d}v =$ „for a at w world v has degree of possibility at least d
- Axiomatics:
  - $KD45(B_a^{\geq d})$ for every a and d
  - $B_a^{\geq d}\,\phi \rightarrow B_a^{\geq d'}\phi$ if $d \geq d'$

KNOWLEDGE VS BELIEF

Can knowledge be defined from belief?

- The antique definition according to Platon (Theaetetus)
  - $K_a\phi = B_a\phi \wedge \phi$ ...
    Problem: knowledge by accident
  - $K_a\phi = B_a\phi \wedge \phi \wedge hasJust(a, \phi)$
    „Knowledge is justified true belief“

- Held to be true for more than 2000 years
- And then comes Gettier
  - Fun fact: idea written on napkin
  - leading to a highly influential 2 page paper (in analytical philosophy) (Gettier 1963)

Gettiers two counterexamples

Scenario 1

- Smith and Jones apply for a job
- Smith believes (justifiably):
  (p) Jones will get the Job &
  John has ten coins in his pocket
- Smith believes also in the entailed assertion:
  (r) The one who gets the job has ten coins in his pocket.
- Coincidence : Smith gets the job and Smith has ten coins in his pocket.
- Smith „knew“ (r) only by chance

Scenario 2

- Smith justifiably believes
  (p) Jones owns a Ford
- Smith also believes in entailed assertion
- (r) = (p or q): Jones owns a Ford, or Brown lives in Barcelona (Though Smith has no justification for q)
- Coincidence: Jones does not own Ford, but Brown lives in Barcelon
- Smith „knew“ (r) only by chance

General idea: decouple justification and truth conditions of propositional content of belief

General remarks

- What is a justification at all?
  - „Solutions“ to Gettier's problem deal with this problem
  - A formal treatmant of justification similar to provability logic: (Artemov 2008)

- Gettier's problem formalized
  - Suppose logic of belief and justification such that
    (*) $\phi \rightarrow \psi \vdash \quad hasJust(a, \phi) \rightarrow hasJust(a, \psi)$
  - Suppose: a wrongly but justifiably believes in p
    $$\neg p \wedge B_a p \wedge hasJust(a, p)$$
  - By M(B_a): $B_a(p \vee q\ ) \wedge B_a(p \vee \neg q)$
  - By (*): $hasJust(a, (p \vee q\ )) \wedge \mathrm{hasJust}(a, (p \vee \neg q))$
  - Hence: $\vDash B_a p \wedge hasJust(a, p) \rightarrow (K_a(p \vee q) \vee K_a(p \vee \neg\, q))$

Relation of knowledge and belief not obvious

- Suppose logic of knowledge and belief defined as
  - $KD45(B_a)$
  - $S5(K_a)$
  - $K_a\phi \rightarrow B_a\phi$
  - $B_a\phi \rightarrow B_aK_a\phi$

- Would entail that $B_a\phi \leftrightarrow K_a\phi$
  (intermediate step: $\neg B_a \neg K_a \phi \rightarrow \neg K_a \neg B_a \phi$)
- Culprit: negative introspection for knowledge
  (Lenzen 1978, Lenzen 1995)

DYNAMICS OF BELIEF

Getting dynamic with beliefs

- How do a's beliefs evolve when a learns that ϕ is true?
- Extend $KD45_n$ by public announcement operator $[\phi!]$
  - What if agent a wrongly believes that p, but $\neg p$ is announced?
  - This is NOT possible in epistemic logic:
    - $\vdash_{S5_n} K_a p \rightarrow p$ (reflexivity)
    - $\vdash_{S5_n PAL} p \leftrightarrow [\neg p!] \perp$ (reduction axiom)
    - $\vdash_{S5_n PAL} K_a p \rightarrow [\neg p!] \perp$
  - In doxastic logic:
    - $B_a p \wedge \neg p$ is satisfiable
    - $\vdash_{KD45_n PAL} p \leftrightarrow [\neg p!] \perp$ (reduction axiom)
    - $B_a p \wedge \neg[\neg p!] \perp$ should be $KD45_n - PAL$ satisfiable

But in doxastic logic dynamics not trivial

- One can show: inconsistent beliefs possible
  $\vdash_{KD45_n PAL} \ (\neg p \wedge B_a p) \rightarrow < \neg p! > B_a \perp$
- Ways out:
  1. Drop seriality
  2. Modify truth condition for announcements

$$M, w \vDash [\phi!]\psi \text{ iff} \begin{pmatrix} M, w \nvDash \ \phi \quad \text{or} \\ M, w \vDash \hat{B}_a\phi \ \text{ and } \ M^{\phi!}, w \vDash \psi \text{ or} \\ M, w \vDash B_a \neg\phi \text{ and } M, w \vDash \psi \end{pmatrix}$$

   - Reduction axiom
     $[\phi!]B_a\psi \leftrightarrow \neg\phi \vee (\hat{B}_a\phi \wedge B_a[\phi!]\psi) \vee (B_a\neg\phi \wedge B_a\psi)$
   - Believe-contravening input is rejected
  3. Integrate belief revision mechanism

Classical theory of Belief Revision

- We partly follow the presentation of Herzig
- For a more comprehensive treatment see also master course „Information Systems CS4130“ at IFIS

- Landmarking „yellow“ paper of Alchourron, Gärdenfors and Makinson (Alchourron et al 1985)

- Beliefs of an ideal agent = set of Boolean formulas $S \subseteq L$ closed under some consequence operator
  - $S \in BS_L$ is called a belief set

AGM takes an internal perspective

- $\phi \in S$ means: ϕ is believed by the agent
- Internal perspective (S is in agent's head)
- Contrast with external perspective:
  - $\phi =$ ``ϕ'' is objectively true
  - Taken in doxastic logic
- But can „internalize"doxastic logic too (Aucher 2008)
  - Distinguished agent Y (for you)
  - $\phi =$ „Y believes that ϕ"
  - Wanted: $\vdash \phi \leftrightarrow B_Y\phi$
  - Abandond inference rule of necessitation:
    $\models B_Y\phi \rightarrow \phi$ but $\not\models B_a(B_Y\phi \rightarrow \phi)$

Coherentism vs foundationalism

- Two general approaches in epistemology
- Foundationalism:
  - All beliefs rest on some basic beliefs (which do not rest by themselves on others, but are assume to be true
  - Some tribute to foundationalism in post AGM-work: Belief bases are (arbitrary not necessarily closed) usually finite sets of sentences
- Coherentism:
  - Beliefs are justified by their relations (consequence, justification..) to other beliefs in a network
  - Usually there is no notion of truth
  - AGM considers closed sets of beliefs based on a consequence operator (logic not based on a semantics)

Types of Belief Change

- L: Set of well-formed formulas (with at least Boolean operators)
- $Cn: 2^L \rightarrow 2^L$ consequence operator (monotonic, idempotent and conclusive)
- B_L: Sets of belief sets = Cn $-$closed sets in 2^L
  - Single inconsistent belief-set = L
- AGM considers three types of operators op all of signature $op: B_L \times L \rightarrow B_L$
  - Expansion: $X + \psi$
  - Contraction: $X - \psi$
  - Revision: $X * \phi$

Types of Belief Change

- $X + \psi$ = expanding X by ψ
  - Result of adding ψ to X without considering inconsistencies
  - Desideratum: $\psi \in X + \psi$ or even $Cn(X \cup \{\psi\}) = X + \psi$
- $X - \psi$ = contracting X by ψ
  - Result of deleting ψ and other sentences such that ψ no longer follows (is contained in the resulting belief set)
  - Desiderata: $\psi \notin X - \psi$; $X - \psi \subseteq X$; ...
- $X * \psi$ = revising X by ψ
  - Result of adding consistently ψ
  - Desiderata: $\psi \in X * \psi$; $X * \psi \neq L$ if $\mathrm{Cn}(\psi) \neq L$

Desiderata captured by AGM postulates

(here for revision)

- (R1) $X * \psi \in BS_L$ (closure)
- (R2) $\psi \in X * \psi$ (success)
- (R3) $X * \psi \subseteq Cn(X \cup \{\psi\})$ (inclusion)
- (R4) If $\neg\psi \notin Cn(X)$ then $Cn(X \cup \{\psi\}) \subseteq X * \psi$ (vacuity)
- (R5) If $Cn(X * \psi) = L$ then $\neg\psi \in Cn(\emptyset)$ (consistency)
- (R6) If $\phi \leftrightarrow \psi \in Cn(\emptyset)$, then $X * \phi = \mathrm{X} * \psi$ (extensionlity)

- (R7) $X * (\phi \wedge \psi) \subseteq Cn((X * \phi) \cup \{\psi\})$ (conjunction 1)
- (R8) If $\neg\beta \notin Cn$, then (conjunction 2)

  $Cn\big((X * \phi) \cup \{\psi\}\big) \subseteq X * (\phi \wedge \psi)$

  (Note: Postulate is not axiom: talks about Cn)

Semantics for AGM

- Postulates generally specify whole classes of operators (exception: expansion)
- How to construct concrete change operators?
- Different design principles
  - Partial meet based on remainder sets (considered here)
  - Orders (epistremic entrenchment)
  - Systems of spheres
  - ...

Remainder Sets: „Maximal Scenarios“

Definition (remainder set)

The remainder set $X \perp \alpha$ of X by α consists of all inclusion-maximal subsets of X not entailing α. The sets in $X \perp \alpha$ are called remainders.

Example

- $\{p, q\} \perp (p \wedge q) = \{\{p\}, \{q\}\}$
- $\{p \vee r, p \vee \neg r, q \wedge s, q \wedge \neg s\} \perp (p \wedge q) =$
  $\{\{p \vee r, p \vee \neg r\}, \{p \vee r, q \wedge s\},$
  $\{p \vee r, q \wedge \neg s\}, \{p \vee \neg r, q \wedge s\}, \{p \vee r, q \wedge \neg s\}$

Selection function

Definition (selection function)

An AGM selection function $\gamma: 2^{B_L} \rightarrow 2^{B_L}$ for X fulfills:

1. If $X \perp \psi \neq \emptyset$, then $\emptyset \neq \gamma(X \perp \alpha) \subseteq X \perp \alpha$
2. $\gamma(\emptyset) = \{X\}$

- As there are many remainders (maximal scenarios) we need to select some of them as possible
- Chooses some remainders (if not empty).

Partial-Meet contraction and revision

Definition

- $X -_\gamma \psi = \bigcap\gamma(X \perp \psi)$ (partial meet contraction)
- $X *_\gamma \psi = Cn((X -_\gamma \neg\psi) \cup \{\psi\})$ (partial meet revision)

Revision operator defined here by so-called Levi-identity from contraction

Theorem (Representation)

An operator $*$ fulfills postulates (R1)-(R6) iff there is a selection function γ sucht that $X * \psi = X *_\gamma \psi$

Notes

- Similar representation result for contraction
- Partial meet revision does not necessarily fulfill (R7) and R(8); need to constrain γ further

AGM: integratio with doxastic logic

- Work of Segerberg (Segerberg 1995, 1996)
  - Modal operators B_a, $[+\psi]$, $[-\psi]$, $[* \psi]$
  - $[* \psi]\phi =$ „ϕ is true after revision by ψ"
- Internal version of doxastic logic (Aucher 2008)
  - Straightforward transfer of AGM representation theorems to multiagent case
- Distinguish several versions of belief (Baltag/Smets 2007, 2009)09
  - Soft beliefs: can be revised
  - Hard beliefs: cannot be revised

GROUP BELIEFS

Group beliefs

- Theory of group beliefs developed in the same way as for group knowledge ...
- $EB_J\phi \coloneqq \bigwedge_{a\in J} B_a\phi$
- $CB_J\phi \coloneqq EB_J\phi \wedge EB_J EB_J\phi \wedge \ \ldots$
- $\mathcal{R}_{CB_J} \coloneqq \left(\bigcup_{a\in J} \mathcal{R}_{B_a}\right)^+$
- Axiomatization of $KD45(B_a)$ with common belief
  - Axiomatics of $KD45(B_a)$
  - Fixed point axiom: $CB_J\phi \leftrightarrow \left(EB_J \wedge EB_J CB_J\phi\right)$
  - Least fixed point inference rule (induction rule)

  $\phi \rightarrow EB_J\,\phi \vdash \ \ EB_J\phi \ \rightarrow CB_J\phi$
- Sound, complete decidable in EXPTIME-c.

Uhhh, a lecture with a hoepfully useful

APPENDIX

References

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Color Convention in this course

- Formulae, when occurring inline
- Newly introduced terminology and definitions
- Important results (observations, theorems) as well as emphasizing some aspects
- Examples are given with standard orange with possibly light orange frame
- Comments and notes
- Algorithms